Egg Problems

William Gasarch-U of MD

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- If an egg **breaks** then you can **cannot** re-use it.

One Egg

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   (Can ignore $+O(1)$ terms.)

Work on in groups.

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   Optimal If g, $2g$, etc and then 1-egg within a gap the worst case is roughly $\frac{n}{g}+g$. This is minimized when $g=n^{1/2}$.

YOU”VE BEEN PUNKED

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- If first drop SPLAT then takes 11 drops.
- If last drop SPLAT then takes 19 drops.
- There should be a way to make the worst case better and the best case worse.

A Trick! A Technique!

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Do Not Need To Have Constant Gap Size!

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$(15 + \cdots + 4)$th Floor. If SPLAT, 3 left, **12+2=14 total**
Sum is 101, so actually works for 101 floors.

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Optimal

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least k with $1 + 2 + \cdots + k \geq n$.
Proof Sketch Any algorithm that deviates from the one we give has to do worse. Formally you would look at the first step where the algorithm differs from ours.

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Can we write $D(e, n)$ in terms of lower values? Yes!

The first move is to drop the egg off of the ith floor.

- If breaks then the problem left is $D(e-1, i)$.
- If does not break then the problem left is $D(e, n-i)$.

$$D(e, n) = 1 + \min_{1 \le i \le n} \max\{D(e-1, i), D(e, n-i)\}.$$