# The Emptier-Filler Game

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There will be a bin with numbers in it.

- If the bin is ever empty then E wins.
- If game goes forever and bin is always nonempty then F wins.

# The Emptier-Filler Game on $\mathbb{N}$

1) F puts a **finite** multiset of $\mathbb{N}$ into the bin.
(e.g., bin has $\{1, 1, 1, 2, 3, 4, 9, 9, 18, 18\}$.

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Which player has the winning strategy? What is that strategy.
**Breakout Rooms!**

# Answer!

E wins!

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**Strategy for E** Keep removing the largest number in the box.

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**What if E plays differently?** One can show that **no matter what E does, she wins!**

**How to prove that?** By an induction on a funky ordering. Won't be doing that.

# The Emptier-Filler Game on Other Orderings

$X$ is any of $\mathbb{Z}$, $\mathbb{Q}^{\geq 0}$, $\mathbb{N}+\mathbb{N}$, $\mathbb{N}+\mathbb{N}+\cdots$, $\mathbb{N}+\mathbb{Z}$, $\mathbb{N}+\mathbb{N}^*$.
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For each of $X=\mathbb{Z}$, $X=\mathbb{Q}$, $X=\mathbb{N}+N$, $X=\mathbb{N}+\mathbb{N}+\cdots$,
$X=\mathbb{N}+\mathbb{Z}$, $X=\mathbb{N}+\mathbb{N}^*$ who wins?
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# Answers!

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# Need a General Theorem

**Question** Let $X$ be a set and $\preceq$ be an ordering on it. Let the $(X, \preceq)$-game be the game as above where we put elements of $X$ in the bin.

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In the following sentence fill in the ???
**E can win the $(X, \preceq)$-game if and only if $(X, \preceq)$ has property ???.**
**Breakout Rooms!**

# Answer!

**Def** $(X, \preceq)$ is **well ordered** if there are NO infinite decreasing sequences.

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**Def** $(X, \preceq)$ is **well ordered** if there are NO infinite decreasing sequences.

**E can win the $(X, \preceq)$-game if and only if $(X, \preceq)$ is well ordered.**