$e$ is Irrational
One Origin of $e$

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(4) Pays out continuously? You have $e$ bills.
Other Definitions of $e$

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$
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This is the one we will use:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots$$
History of $e$ Irrational, $e$ Transcendental

1. Euler showed $e$ was irrational in 1737 using continued fractions.
2. Fourier gave an elementary proof in 1815 which we will present.
3. Liouville proved that a particular contrived number was transcendental in 1851.
4. Hermite proved $e$ is transcendental in 1873. 1st non-contrived number to be proven transcendental.
5. Cantor proved that most numbers are transcendental in 1874, and gave a method for constructing some.
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Proof that e is Irrational
Warmup: Proof that $e \notin \mathbb{N}$

$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots = 2.5 + \sum_{i=3}^{\infty} \frac{1}{i!}.$

For $i \geq 3$, $\frac{1}{i!} < \frac{1}{2^{i-1}}$.

$\sum_{i=3}^{\infty} \frac{1}{i!} < \sum_{i=3}^{\infty} \frac{1}{2^{i-1}} = \frac{1}{2}.$

Hence $e = 2.5 + \sum_{i=3}^{\infty} \frac{1}{i!} < 2.5 + 0.5 = 3$.

Hence $2 < e < 3$ so $e \notin \mathbb{N}$. 
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Assume $e$ is rational. So $\exists a, b \in \mathbb{N}$ such that $e = \frac{a}{b}$.

Let $n \in \mathbb{N}$ be named later.

eb = a, so $bn! e = n! a \in \mathbb{N}$. 

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$$bn!e = bn! \left( \left( 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \right) + \left( \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots \right) \right)$$
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$$bn!e = b\left(C + \frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots\right) \in \mathbb{N}.$$
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\]
For large $n$ we can approximate it very well by

\[
\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots
\]

\[
= \frac{1}{n} \left( 1 + \frac{1}{n} + \frac{1}{n^2} + \cdots + \right) \frac{1}{n} \frac{1}{1 - (1/n)} = \frac{1}{n - 1}
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Take $n$ big enough and this cannot happen. Contradiction!