Review For Final

250H

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Proof: First take the contrapositive

If $x \not\equiv 0 \pmod 5$ then $x^2 \not\equiv 0 \pmod 5$.

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Proof: First take the contrapositive

If $x \not\equiv 0 \pmod 5$ then $x^2 \not\equiv 0 \pmod 5$.

We do this by cases. All $\equiv$ are mod 5.

If $x \equiv 1$ then $x^2 \equiv 1^2 \equiv 1 \not\equiv 0$

If $x \equiv 2$ then $x^2 \equiv 2^2 \equiv 4 \not\equiv 0$

If $x \equiv 3$ then $x^2 \equiv 3^2 \equiv 9 \equiv 4 \not\equiv 0$

If $x \equiv 4$ then $x^2 \equiv 4^2 \equiv 16 \equiv 1 \not\equiv 0$

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Theorem: $\sqrt{5} \notin Q$.

Proof: Assume, by way of contradiction, that $\sqrt{5} \in Q$. Hence there exists a, b IN LOWEST TERMS such that

$\sqrt{5} = \frac{a}{b}$

$5 = \frac{a^2}{b^2}$

$5b^2 = a^2$

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So $a^2 \equiv 0 \pmod 5$. By Lemma $a \equiv 0 \pmod 5$. Let $a = 5c$

$5b^2 = a^2$ is now

$5b^2 = (5c)^2 = 25c^2$

$b^2 = 5c^2$.

So $b^2 \equiv 0 \pmod 5$. By Lemma $b \equiv 0 \pmod 5$.

We now have that a and b both have a factor of 5. Hence a, b are NOT in lowest terms. This CONTRADICTS that a, b are not in lowest terms.

Prove that $\sqrt{5} \notin Q$ using unique factorization.

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We FACTOR a, b:

$a = p_1^{a_1} \cdots p_L^{a_L}$

so $a^2 = p_1^{2a_1} \cdots p_L^{2a_L}$

$b = p_1^{b_1} \cdots p_L^{b_L}$

so $b^2 = p_1^{2b_1} \cdots p_L^{2b_L}$

(NOTE- some of the a_i's and b_i's could be 0.)

Prove that $\sqrt{5} \notin Q$ using unique factorization.

$5b^2 = a^2$

so

$$5p_1^{2b_1} \cdots p_L^{2b_L} = p_1^{2a_1} \cdots p_L^{2a_L}$$

Be reordering let $p_1 = 5$.

The number of 5's on the LHS is $2b_1 + 1$.

The number of 5's on the RHS is $2a_1$.

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Hence

$2b_1 + 1 = 2a_1$

$1 = 0$

CONTRADICTION.

(20 points) For the following sentences find both (a) an infinite domain where it is true, and (b) an infinite domain where it is false. All domains should be subsets of R.

$(\forall x)(\exists y)[x = y^2]$ but DO NOT use R or any closed or open or clopen interval.

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Let

$$D_0 = Z$$

$$D_1 = D_0 \cup \{\sqrt{x} \mid x \in D_0\}$$

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Let

$D_0 = Z$

$D_1 = D_0 \cup \{\sqrt{x} \mid x \in D_0\}$

For all $i \geq 2$

$D_i = D_0 \cup \{\sqrt{x} \mid x \in D_0 \cup \cdots \cup D_{i-1}\}$

Now let

$$D = D_0 \cup D_1 \cup \cdots$$

Every number in D_i has a square root in D_{i+1}, hence every element of D has a square root in D.

(30 points — 10 each) Show that the following sets are uncountable

(a) The set of functions from N to N that are strictly increasing. (That means that, for all $x, y \in \mathsf{N}$, if $x < y$ then $f(x) < f(y)$.)

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1) Assume, BWOC, that $f_1, f_2, f_3, \ldots$ is the set of ALL increasing functions.

We CONSTRUCT a function that is increasing but is not one of $f_1, f_2, \ldots$.

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The function is F. We will make sure that $(\forall i)[F(i) \neq f_i(i)]$ and hence $(\forall i)[F \neq f_i]$. The trick will be to make sure that F is increasing.

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$F(0) = f_0(0) + 1$

If $i \geq 1$ then let

$$F(i) = \max\{F(0), \ldots, F(i-1), f_i(i)\} + 1$$

F is clearly increasing AND, for all i, $F(i) \neq f_i(i)$.

The set of functions from N to PRIMES.

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$$F(i) = \text{ the next prime after } f_i(i)$$

F clearly only takes on prime values AND, for all i, $F(i) \neq f_i(i)$.

Find A, B, C, D by constructive induction.

$$(\forall n \geq 0, n \in \mathsf{N}) \left[\sum_{i=1}^{n} i \times 2^i = An2^n + B2^n + Cn + D \right].$$

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2) By Constructive Induction.

Base Case: $n = 0$ yields $B + D = 0$

IH:

$$\sum_{i=1}^{n} i \times 2^i = An2^n + B2^n + Cn + D.$$

$$An2^n+B2^n+Cn+D+(n+1)2^{n+1} = An2^{n+1}+A2^{n+1}+B2^{n+1}+C(n+1)+D$$

$$An2^n + B2^n + Cn + (n+1)2^{n+1} = An2^{n+1} + A2^{n+1} + B2^{n+1} + C(n+1)$$

$$An2^n + B2^n + (n+1)2^{n+1} = An2^{n+1} + A2^{n+1} + B2^{n+1} + C$$

$$(n+1)2^{n+1} = An2^n + A2^{n+1} + B2^n + C$$

$$(2n)2^n + 2(2^n)) = An2^n + (2A+B)2^n + C$$

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$A = 2$, $B = -2$, $C = 0$.

Since $B + D = 0$, $D = 2$ So we get

$$\sum_{i=1}^{n} i \times 2^i = 2n2^n + -2 \times 2^n + 2.$$

(25 points) Use a combinatorial argument (NOT algebraic, NOT by induction) to show that if $S = a + b + c$ then

$$\frac{S!}{a!b!c!} = \frac{(S-1)!}{(a-1)!b!c!} + \frac{(S-1)!}{a!(b-1)!c!} + \frac{(S-1)!}{a!b!(c-1)!}$$

The number of ways of assigning a students to WG, b students to K, c students to Jtwitty is

$$\frac{S!}{a!b!c!}.$$

We can break this up into three disjoint sets:

Let Alice be a student in the class.

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$$\frac{(S-1)!}{(a-1)!b!c!}.$$

The number of ways they can do this where Jtwitty drives Alice is

$$\frac{(S-1)!}{a!b!(c-1)!}.$$

Add them up for the proof!

(10 points) WG, Jtwitty, and K are taking the class on a field trip to the Combinatorics Museum! There are 32 students in the class

WG will drive 18 of them.

Jtwitty will drive 7 of them.

K will drive 7 of them.

How many ways can the students choose which cars they want to be in?

1a) We do it two ways

18 students are chosen for WG, so that $\binom{32}{18}$. Note that there are 14 students left.

7 students of those left are chosen for Jtwitty, so that $\binom{14}{7}$.

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18 students are chosen for WG, so that $\binom{32}{18}$. Note that there are 14 students left.

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The rest go to K.

So the answer is $\binom{32}{18} \times \binom{14}{7}$.

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we then want to NOT CARE about the order WITHIN the first 18, WITHIN the next 7, WITHIN the next 7. So thats directly $\frac{32}{18!7!7!}$

(b) (15 points) Generalize the problem as follows. $A_1, \dots, A_n$ are taking the class on a field trip! There are S students in the class.

A_1 will drive a_1 of them.

$\vdots$

A_n will drive a_n of them.

(Note that $a_1 + \cdots + a_n = S$.)

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$$\binom{S!}{a!b!c!}$$