Review For Final

250H
Prove that \( \sqrt{5} \notin \mathbb{Q} \) using the mod method. (Hint: First prove a lemma about mods.)
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*Lemma:* If $x^2 \equiv 0 \pmod{5}$ then $x \equiv 0 \pmod{5}$. 
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**Lemma:** If $x^2 \equiv 0 \pmod{5}$ then $x \equiv 0 \pmod{5}$.

**Proof:** First take the contrapositive

If $x \not\equiv 0 \pmod{5}$ then $x^2 \not\equiv 0 \pmod{5}$. 
Prove that $\sqrt{5} \notin \mathbb{Q}$ using the mod method. (Hint: First prove a lemma about mods.)

**Lemma:** If $x^2 \equiv 0 \pmod{5}$ then $x \equiv 0 \pmod{5}$.

**Proof:** First take the contrapositive

If $x \not\equiv 0 \pmod{5}$ then $x^2 \not\equiv 0 \pmod{5}$.

We do this by cases. All $\equiv$ are mod 5.

If $x \equiv 1$ then $x^2 \equiv 1^2 \equiv 1 \not\equiv 0$

If $x \equiv 2$ then $x^2 \equiv 2^2 \equiv 4 \not\equiv 0$

If $x \equiv 3$ then $x^2 \equiv 3^2 \equiv 9 \equiv 4 \not\equiv 0$

If $x \equiv 4$ then $x^2 \equiv 4^2 \equiv 16 \equiv 1 \not\equiv 0$
Prove that $\sqrt{5} \notin \mathbb{Q}$ using the mod method. (Hint: First prove a lemma about mods.)

**Theorem:** $\sqrt{5} \notin \mathbb{Q}$.

**Proof:** Assume, by way of contradiction, that $\sqrt{5} \in \mathbb{Q}$. Hence there exists $a, b$ IN LOWEST TERMS such that

$$\sqrt{5} = \frac{a}{b}$$
$$5 = \frac{a^2}{b^2}$$
$$5b^2 = a^2$$
Prove that $\sqrt{5} \notin Q$ using the mod method. (Hint: First prove a lemma about mods.)

So $a^2 \equiv 0 \pmod{5}$. By Lemma $a \equiv 0 \pmod{5}$. Let $a = 5c$

$5b^2 = a^2$ is now

$5b^2 = (5c)^2 = 25c^2$

$b^2 = 5c^2$.

So $b^2 \equiv 0 \pmod{5}$. By Lemma $b \equiv 0 \pmod{5}$.

We now have that $a$ and $b$ both have a factor of 5. Hence $a, b$ are NOT in lowest terms. This CONTRADICTS that $a, b$ are not in lowest terms.
Prove that $\sqrt{5} \notin \mathbb{Q}$ using unique factorization.

Theorem: $\sqrt{5} \notin \mathbb{Q}$.

Proof: Assume, by way of contradiction, that $\sqrt{5} \in \mathbb{Q}$. Hence there exists $a, b$ such that:

$$\sqrt{5} = \frac{a}{b}$$
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Prove that $\sqrt{5} \notin \mathbb{Q}$ using unique factorization.

$$5b^2 = a^2$$

We FACTOR $a, b$:

$$a = p_1^{a_1} \cdots p_L^{a_L}$$

so

$$a^2 = p_1^{2a_1} \cdots p_L^{2a_L}$$

$$b = p_1^{b_1} \cdots p_L^{b_L}$$

so

$$b^2 = p_1^{2b_1} \cdots p_L^{2b_L}$$

(NOTE- some of the $a_i$’s and $b_i$’s could be 0.)
Prove that $\sqrt{5} \notin \mathbb{Q}$ using unique factorization.

$5b^2 = a^2$

so

$$5p_1^{2b_1} \cdots p_L^{2b_L} = p_1^{2a_1} \cdots p_L^{2a_L}$$

By reordering let $p_1 = 5$.

The number of 5’s on the LHS is $2b_1 + 1$.

The number of 5’s on the RHS is $2a_1$. 
Prove that $\sqrt{5} \notin Q$ using unique factorization.

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Be reordering let $p_1 = 5$.

The number of 5’s on the LHS is $2b_1 + 1$.

The number of 5’s on the RHS is $2a_1$.

Hence

$$2b_1 + 1 = 2a_1$$

$$1 = 0$$

CONTRADICTION.
(20 points) For the following sentences find both (a) an infinite domain where it is true, and (b) an infinite domain where it is false. All domains should be subsets of $R$.

$(\forall x)(\exists y)[x = y^2]$ but DO NOT use $R$ or any closed or open or clopen interval.
$(\forall x)(\exists y)[x = y^2]$ but DO NOT use $R$ or any closed or open or clopen interval.

Let

\[ D_0 = \mathbb{Z} \]

\[ D_1 = D_0 \cup \{ \sqrt{x} \mid x \in D_0 \} \]}
$(\forall x)(\exists y)[x = y^2]$ but DO NOT use $R$ or any closed or open or clopen interval.

Let

$D_0 = \mathbb{Z}$

$D_1 = D_0 \cup \{\sqrt{x} \mid x \in D_0\}$

For all $i \geq 2$

$D_i = D_0 \cup \{\sqrt{x} \mid x \in D_0 \cup \cdots \cup D_{i-1}\}$

Now let

$$D = D_0 \cup D_1 \cup \cdots$$

Every number in $D_i$ has a square root in $D_{i+1}$, hence every element of $D$ has a square root in $D$. 
(30 points — 10 each) Show that the following sets are uncountable

(a) The set of functions from \( N \) to \( N \) that are strictly increasing. (That means that, for all \( x, y \in N \), if \( x < y \) then \( f(x) < f(y) \)).
(30 points — 10 each) Show that the following sets are uncountable

(a) The set of functions from \( \mathbb{N} \) to \( \mathbb{N} \) that are strictly increasing. (That means that, for all \( x, y \in \mathbb{N} \), if \( x < y \) then \( f(x) < f(y) \).)

1) Assume, BWOC, that \( f_1, f_2, f_3, \ldots \) is the set of ALL increasing functions.

We CONSTRUCT a function that is increasing but is not one of \( f_1, f_2, \ldots \).
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1) Assume, BWOC, that \( f_1, f_2, f_3, \ldots \) is the set of ALL increasing functions.

We CONSTRUCT a function that is increasing but is not one of \( f_1, f_2, \ldots \). The function is \( F \). We will make sure that \( (\forall i)[F(i) \neq f_i(i)] \) and hence \( (\forall i)[F \neq f_i] \). The trick will be to make sure that \( F \) is increasing.

\[ F(0) = f_0(0) + 1 \]
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\[
F(0) = f_0(0) + 1
\]

If \( i \geq 1 \) then let

\[
F(i) = \max\{F(0), \ldots, F(i-1), f_i(i)\} + 1
\]

\( F \) is clearly increasing AND, for all \( i \), \( F(i) \neq f_i(i) \).
The set of functions from $\mathbb{N}$ to PRIMES.

2) Assume, BWOC, that $f_1, f_2, f_3, \ldots$ is the set of ALL functions from $\mathbb{N}$ to the PRIMES.
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We CONSTRUCT a function that is from $\mathbb{N}$ to PRIMES but is not one of $f_1, f_2, \ldots$. The trick will be to make sure that $F$ only takes on values in the primes.

$$F(i) = \text{the next prime after } f_i(i)$$

$F$ clearly only takes on prime values AND, for all $i$, $F(i) \neq f_i(i)$. 
Find $A, B, C, D$ by constructive induction.

$(\forall n \geq 0, n \in \mathbb{N}) \left[ \sum_{i=1}^{n} i \times 2^i = A n 2^n + B 2^n + C n + D \right]$. 
Find $A, B, C, D$ by constructive induction.

$(\forall n \geq 0, n \in \mathbb{N}) \left[ \sum_{i=1}^{n} i \times 2^i = A2^n + B2^n + Cn + D \right]$. 

2) By Constructive Induction.

**Base Case:** $n = 0$ yields $B + D = 0$

**IH:**

$$\sum_{i=1}^{n} i \times 2^i = A2^n + B2^n + Cn + D.$$
\[ An2^n + B2^n + Cn + D + (n+1)2^{n+1} = An2^{n+1} + A2^{n+1} + B2^{n+1} + C(n+1) + D \]

\[ An2^n + B2^n + Cn + (n+1)2^{n+1} = An2^{n+1} + A2^{n+1} + B2^{n+1} + C(n+1) \]

\[ An2^n + B2^n + (n + 1)2^{n+1} = An2^{n+1} + A2^{n+1} + B2^{n+1} + C \]

\[ (n + 1)2^{n+1} = An2^n + A2^{n+1} + B2^n + C \]

\[ (2n)2^n + 2(2^n)) = An2^n + (2A + B)2^n + C \]
\[(n + 1)2^{n+1} = An2^n + A2^{n+1} + B2^n + C\]

\[(2n)2^n + 2(2^n)) = An2^n + (2A + B)2^n + C\]

\[A = 2, \quad B = -2, \quad C = 0.\]

Since \(B + C = 0\), \(D = 2\) So we get

\[\sum_{i=1}^{n} i \times 2^i = 2n2^n - 2 \times 2^n + 2.\]
(25 points) Use a combinatorial argument (NOT algebraic, NOT by induction) to show that if \( S = a + b + c \) then

\[
\frac{S!}{a!b!c!} = \frac{(S - 1)!}{(a - 1)!b!c!} + \frac{(S - 1)!}{a!(b - 1)!c!} + \frac{(S - 1)!}{a!b!(c - 1)!}
\]

The number of ways of assigning \( a \) students to WG, \( b \) students to K, \( c \) students to Jtwitty is

\[
\frac{S!}{a!b!c!}.
\]

We can break this up into three disjoint sets:

Let Alice be a student in the class.
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The number of ways they can do this where WG drives Alice is
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Let Alice be a student in the class.

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The number of ways they can do this where K drives Alice is

\[
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\]
The number of ways they can do this where K drives Alice is

\[
\frac{(S - 1)!}{(a - 1)!b!c!}.
\]

The number of ways they can do this where Jtwitty drives Alice is

\[
\frac{(S - 1)!}{a!b!(c - 1)!}.
\]

Add them up for the proof!
(10 points) WG, Jtwitty, and K are taking the class on a field trip to the Combinatorics Museum! There are 32 students in the class. WG will drive 18 of them. Jtwitty will drive 7 of them. K will drive 7 of them. How many ways can the students choose which cars they want to be in?
1a) We do it two ways

18 students are chosen for WG, so that \( \binom{32}{18} \). Note that there are 14 students left.

7 students of those left are chosen for Jtwitty, so that \( \binom{14}{7} \).
1a) We do it two ways

18 students are chosen for WG, so that \( \binom{32}{18} \). Note that there are 14 students left.

7 students of those left are chosen for Jtwitty, so that \( \binom{14}{7} \).

The rest go to K.

So the answer is \( \binom{32}{18} \times \binom{14}{7} \).

This is a fine answer; however note that it is also \( \frac{32}{18!7!7!} \) which leads to the second method.
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we then want to NOT CARE about the order WITHIN the first 18, WITHIN the next 7, WITHIN the next 7. So thats directly $\frac{32}{18!7!7!}$
(b) (15 points) Generalize the problem as follows. $A_1, \ldots, A_n$ are taking the class on a field trip! There are $S$ students in the class. $A_1$ will drive $a_1$ of them.

$A_n$ will drive $a_n$ of them.

(Note that $a_1 + \cdots + a_n = S$.)

How many ways can the students choose which cars they want to be in?
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$A_n$ will drive $a_n$ of them.

(Note that $a_1 + \cdots + a_n = S$.)

How many ways can the students choose which cars they want to be in?

\[
\left( \frac{S!}{a_1!b_1!c_1!} \right)
\]