The Forehead Game

Exposition by William Gasarch

May 3, 2021

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STUDENTS WORK IN GROUPS TO BEAT $n+1$ OR SHOW YOU CAN"T

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   Bob knows a_i's and c_i's so he can compute
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   $s = 1^{n/2}$: Bob says (MAYBE,z). $s \neq 1^{n/2}$: Bob says NO.
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   Carol knows the carry bit z so she can compute
   $a_0 \cdots a_{n/2} + b_0 \cdots b_{n/2} + c_0 \cdots c_{n/2} + z = t$
   $t = 1^{n/2}$: Carol says YES. $t \neq 1^{n/2}$: Carol says NO.

Four People

Alice is A, Bob is B, Carol is C, Donna is D.

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STUDENTS WORK IN GROUPS TO EITHER DO BETTER THAN $n + 1$ OR SHOW YOU CAN"T

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3. Carol can add first $1/3$ of the bits, sees if its $1^{n/3}$, if its not say NO, if it is say MAYBE and the carry bit.

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3. Carol can add first $1/3$ of the bits, sees if its $1^{n/3}$, if its not say NO, if it is say MAYBE and the carry bit.
4. Bob can add second $1/3$ of the bits along with the carry bit, sees if its $1^{n/3}$, if its not say NO, if it is say MAYBE and the carry bit.

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5. Bob can add third $1/3$ of the bits along with the carry bit, sees if its $1^{n/3}$, if its not say NO, if it is say YES.

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4. Can do in $\frac{n}{k-1} + O(1)$ bits, similar to the $k = 3, 4$ cases.
5. Caveat: The $O(1)$ term is really $O(k)$ which matters if k is a function of n.

Final Vote and the Answer

Lets go back to 3 people. We know we can do $\frac{n}{2} + O(1)$.

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3. There is an $O\left(\frac{n}{\log n}\right)$ protocol, optimal UNKNOWN.

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3. There is an $O\left(\frac{n}{\log n}\right)$ protocol, optimal UNKNOWN.
4. There exists an $O(n^{1-\delta})$ protocol and it is roughly optimal.
5. There exists an $O(n^{1-\delta})$ protocol, optimal UNKNOWN.

VOTE!

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3 people:

- Chandra-Furst-Lipton (1983): there is an $O(n^{1/2})$ protocol; lower bound $\Omega(1)$.

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- Chandra-Furst-Lipton (1983): there is an $O(n^{1/2})$ protocol; lower bound $\Omega(1)$.
- Gasarch (2006): Lower Bound $\Omega(\log\log n)$.
- Nothing else is known.

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- Gasarch 2006: there is an $O(n^{1/(\log_2(k-1)})$ protocol. (A more careful analysis of Chandra-Furst-Lipton protocol.)
- Chandra-Furst-Lipton, lower bound $\Omega(1)$.
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Open Similar questions for 4 people, 5 people, etc.