The Hat Check Problem
Hat Check Problem

▶ $n$ people give their hats to a hat check person.
▶ The hat check person gives people their hats RANDOMLY.
▶ What is Prob NOBODY gets their correct hat?
What are your Intuitions?

1. Do you think that as $n$ gets large the prob that nobody gets their correct hat goes up or goes down or oscillates?

2. It limits to a value $v$.
   2.1 $0 < v < \frac{1}{4}$
   2.2 $\frac{1}{4} \leq v < \frac{1}{2}$
   2.3 $\frac{1}{2} \leq v < \frac{3}{4}$
   2.4 $\frac{3}{4} \leq v < 1$.

Will answer at the end of this slide packet.
What are your Intuitions?

1. Do you think that as \( n \) gets large the prob that nobody gets their correct hat goes up or goes down or oscillates?
What are your Intuitions?

1. Do you think that as $n$ gets large the prob that nobody gets their correct hat goes up or goes down or oscillates?

2. It limits to a value $v$. Vote!
   - 2.1 $0 < v < \frac{1}{4}$.
   - 2.2 $\frac{1}{4} \leq v < \frac{1}{2}$.
   - 2.3 $\frac{1}{2} \leq v < \frac{3}{4}$.
   - 2.4 $\frac{3}{4} \leq v < 1$.

Will answer at the end of this slide packet.
The two people are named 1, 2. The hats are labeled 1, 2.

Number of ways the people can get their hats: $2! = 2$. In 1 of those nobody gets their hat back. So Prob is $\frac{1}{2}$. 

$n = 2$
P(i) will be prob that person i gets their hat back
P(i,j) will be prob that persons i and j get their hat back
etc.

ALSO
P(some) is Prob someone has the right hat.
The three people are named 1, 2, 3.
The hats are labeled 1, 2, 3.
The three people are named 1, 2, 3. The hats are labeled 1, 2, 3.

**KEY:** We do prob that SOMEONE DOES get right hat.

\[ P(1) = \frac{1}{3} \]
\[ P(2) = \frac{1}{3} \]
\[ P(3) = \frac{1}{3} \]

So prob that SOMEONE gets the right hat is \( \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1 \). REALLY?
\( n = 3 \) Continued

\[
P(\text{some}) = P(1) + P(2) + P(3) - P(1,2) - P(1,3) - P(2,3) + P(1,2,3) \\
= 3P(1 \text{ right}) - 3P(1,2 \text{ right}) + P(1,2,3)
\]

\[
P(1) = \frac{1}{3} \\
P(1,2) = \frac{1}{6} \\
P(1,2,3)=\frac{1}{6}
\]

So Prob is \( 3 \times \frac{1}{3} - 3 \times \frac{1}{6} - \frac{1}{6} = \frac{2}{3} \).

So Prob NOBODY gets right hat is \( 1 - \frac{2}{3} = \frac{1}{3} \).
\( n = 4 \)

\[
P(\text{some}) = P(1) + P(2) + P(3) + P(4) \\
- (P(1,2)+P(1,3)+P(1,4)+P(2,3)+ P(2,4) + P(3,4)) \\
+ (P(1,2,3)+P(1,2,4)+P(1,3,4)+P(2,3,4) \\
- P(1,2,3,4)
\]
\textbf{\( n = 4 \)}

\[ P(\text{some}) = P(1) + P(2) + P(3) + P(4) \]
\[ - (P(1,2) + P(1,3) + P(1,4) + P(2,3) + P(2,4) + P(3,4)) \]
\[ + (P(1,2,3) + P(1,2,4) + P(1,3,4) + P(2,3,4) - P(1,2,3,4)) \]
\[ - P(1,2,3,4) \]

\textbf{EASIER:}

\[ P(\text{some}) = \binom{4}{1} P(1) - \binom{4}{2} P(1,2) + \binom{4}{3} P(1,2,3) - \binom{4}{4} P(1,2,3,4) \]
\( n = 4 \)

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+ (P(1,2,3) + P(1,2,4) + P(1,3,4) + P(2,3,4)
- P(1,2,3,4))
- P(1,2,3,4)
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\]
\[
P(1) = \frac{3!}{4!}
\]
\[
P(1,2) = \frac{2!}{4!}
\]
\[
P(1,2,3) = \frac{1!}{4!}
\]
\[
P(1,2,3,4) = \frac{1!}{4!}
\]
\( n = 4 \)

\[
P(\text{some}) = P(1) + P(2) + P(3) + P(4) \\
- (P(1,2)+P(1,3)+P(1,4)+P(2,3)+ P(2,4) + P(3,4)) \\
+ (P(1,2,3)+P(1,2,4)+P(1,3,4)+P(2,3,4) \\
- P(1,2,3,4))
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P(1,2,3,4) = \frac{1!}{4!}
\]

\[
P(\text{some}) = \frac{4!}{1!3!4!} \cdot \frac{3!}{4!} - \frac{4!}{2!2!4!} \cdot \frac{2!}{4!} + \frac{4!}{3!1!4!} \cdot \frac{1!}{4!} - \frac{4!}{0!4!4!}
\]

\[
= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} = \frac{15}{24} = \frac{5}{8}
\]
Table of Results So Far

<table>
<thead>
<tr>
<th>$n$</th>
<th>Prob nobody gets their hat back</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\frac{1}{2} = 0.5$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{3} = 0.33\ldots$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{5}{8} = 0.625$</td>
</tr>
</tbody>
</table>

So far it looks like it osculating, but not much evidence.
General Case

\[ P(\text{some}) = \binom{n}{1} P(1) - \binom{n}{2} P(1,2) + \binom{n}{3} P(1,2,3) \cdots \pm \binom{n}{n} P(1,\ldots,n) \]
General Case

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\[ P(1) = \frac{(n-1)!}{n!} \]

\[ P(1,2) = \frac{(n-2)!}{n!} \]

\[ P(1,2,3) = \frac{(n-3)!}{n!} \]

etc.

\[ P(1,\ldots, n) = \frac{1!}{n!} \]
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etc.

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\[ P(\text{some}) = \]

\[ \frac{n!}{1!(n-1)!} \cdot \frac{(n-1)!}{n!} - \frac{n!}{2!(n-2)!} \cdot \frac{(n-2)!}{n!} \pm \frac{1}{n!} \]

\[ = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} \pm \frac{1}{n!} \]
Pass to the Infinite Sum

We approximate

\[= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} \pm \frac{1}{n!}\]

Is this a good approximation? Discuss.

Yes

The error term is something like \(\frac{1}{(n+1)!} - \frac{1}{(n-1)!} + \cdots \leq \frac{1}{(n+1)!}\)

Even this upper bound is an overestimate.

Upshot

Approximation is very good.
Pass to the Infinite Sum

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$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} \pm \frac{1}{n!}$$

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Is this a good approximation? Discuss.

**Yes** The error term is something like

\[\frac{1}{(n + 1)!} - \frac{1}{(n - 1)!} + \cdots \leq \frac{1}{(n + 1)!}\]

Even this upper bound is an overestimate.

**Upshot** Approximation is very good.
How to Sum the Series

\[ P(\text{some}) = \]

\[ = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} \pm \cdots \]

So \( P(\text{nobody has the right hat}) = \)

\[ \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cdots \]

Do you know how to sum this series?
How to Sum the Series

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So \( P(\text{nobody has the right hat}) = \)

\[ \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cdots \]

Do you know how to sum this series? Recall that

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \]

\[ \frac{1}{e} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots = \frac{1}{2!} - \frac{1}{3!} + \cdots \]

SO final answer:

Prob nobody has right hat is \( \sim \frac{1}{e} \).
Reflection

If 1000 people check their hats and get the back randomly, prob nobody gets their hat is very close to \(\frac{1}{e}\).
If 1000 people check their hats and get the back randomly, prob nobody gets their hat is very close to $\frac{1}{e}$.

If 1,000,000 people check their hats and get the back randomly, prob nobody gets their hat is very close to $\frac{1}{e}$. 
Reflection

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If 1,000,000 people check their hats and get the back randomly, prob nobody gets their hat is very close to \( \frac{1}{e} \).

1. Are you surprised the answer is \( \frac{1}{e} \)?
Reflection

If 1000 people check their hats and get the back randomly, prob nobody gets their hat is very close to $\frac{1}{e}$.
If 1,000,000 people check their hats and get the back randomly, prob nobody gets their hat is very close to $\frac{1}{e}$.

1. Are you surprised the answer is $\frac{1}{e}$? Surprised at how big this is?
If 1000 people check their hats and get the back randomly, prob **nobody** gets their hat is **very close** to \( \frac{1}{e} \).
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   Surprised at how big this is?
   How small this is?
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1. Are you surprised the answer is $\frac{1}{e}$?
   Surprised at how big this is?
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   How nice this is?
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1. Are you surprised the answer is $\frac{1}{e}$?
   Surprised at how big this is?
   How small this is?
   How nice this is?

2. Are you surprised that for $n$ large its very stable at around $\frac{1}{e}$?