Strong Induction and Inequalities
In the Strong Induction Slide Packet we studied the recurrence

\[ a_n = \begin{cases} 
1 & \text{if } n = 0 \\
8 & \text{if } n = 1 \\
a_{n-1} + 2a_{n-2} & \text{if } n \geq 2 
\end{cases} \]  

(1)

The solution is

\[ a_n = 3 \cdot 2^n + 2(-1)^{n+1}. \]
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Nice Recurrences

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NICE SOLUTION!
Sort-of Nice Recurrences

In recitation you saw that the recurrence

\[ a_n = \begin{cases} 
1 & \text{if } n = 0 \\
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\end{cases} \tag{2} \]

This solution is

\( a_n = \frac{\sqrt{5}}{2} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right) \)

(1) YEAH- an exact formula

(2) BOO- hard to use numerically.

(3) YEAH- Bill thinks it's Jawesome (Jaw-Dropping Awesome).

(4) BOO- Emily thinks it's gross.

(5) It could be worse.

Next Slide.
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This solution is

\[ a_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right) \]
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Not Nice

\[ a_n = \begin{cases} 
1 & \text{if } n = 0 \\
2 & \text{if } n = 1 \\
3 & \text{if } n = 2 \\
a_{n-1} + 11a_{n-2} + 13a_{n-3} & \text{if } n \geq 2 
\end{cases} \] (3)

Does this have a closed form solution? VOTE

YES, and it only involves integers

YES, but it involves irrationals

NO.

The answer is YES, but it involves irrationals.

Rather than bother with an exact solution, we will prove an upper bound that is nice.
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\end{cases} \)  

Does this have a closed form solution? VOTE
YES, and it only involves integers
YES, but it involves irrationals
NO.

The answer is
YES, but it involves irrationals.

Rather than bother with an exact solution, we will prove an UPPER BOUND that IS nice.
Upper Bound

\[ a_n = \begin{cases} 
1 & \text{if } n = 0 \\
2 & \text{if } n = 1 \\
3 & \text{if } n = 2 \\
a_{n-1} + 11a_{n-2} + 13a_{n-3} & \text{if } n \geq 2 
\end{cases} \] (4)

**Thm** \((\forall n)[a_n \leq 5^n]\)
Upper Bound

\[ a_n = \begin{cases} 
1 & \text{if } n = 0 \\
2 & \text{if } n = 1 \\
3 & \text{if } n = 2 \\
\ a_{n-1} + 11a_{n-2} + 13a_{n-3} & \text{if } n \geq 2
\end{cases} \]  \hspace{2cm} (4)

**Thm** \((\forall n)[a_n \leq 5^n]\)

**Base Case** \(a_0 = 1 \leq 5^0 = 1\) YES. Also \(a_1 = 2 \leq 5^1 = 5\).
Upper Bound

\[ a_n = \begin{cases} 
1 & \text{if } n = 0 \\
2 & \text{if } n = 1 \\
3 & \text{if } n = 2 \\
 a_{n-1} + 11a_{n-2} + 13a_{n-3} & \text{if } n \geq 2 
\end{cases} \]  \hspace{1cm} (4)

**Thm** (\(\forall n\))[\(a_n \leq 5^n\)]

**Base Case** \(a_0 = 1 \leq 5^0 = 1\) YES. Also \(a_1 = 2 \leq 5^1 = 5\).

**IH** \(n \geq 2\). (\(\forall n' < n\))[\(a_{n'} \leq 5^{n'}\)]. In particular

- \(a_{n-1} \leq 5^{n-1}\),
- \(a_{n-2} \leq 5^{n-2}\),
- \(a_{n-3} \leq 5^{n-3}\).
Upper Bound

\[
a_n = \begin{cases} 
  1 & \text{if } n = 0 \\
  2 & \text{if } n = 1 \\
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  a_{n-1} + 11a_{n-2} + 13a_{n-3} & \text{if } n \geq 2 
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**Base Case** \(a_0 = 1 \leq 5^0 = 1\) YES. Also \(a_1 = 2 \leq 5^1 = 5\).

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\begin{align*}
  a_{n-1} & \leq 5^{n-1}, \\
  a_{n-2} & \leq 5^{n-2}, \\
  a_{n-3} & \leq 5^{n-3}.
\end{align*}
\]

Finish on next slide.
Upper Bound

Recall $a_n = a_{n-1} + 11a_{n-2} + 13a_{n-3}$.

Recall

$a_{n-1} \leq 5^{n-1}$  $a_{n-2} \leq 5^{n-2}$  $a_{n-3} \leq 5^{n-3}$.
Upper Bound

Recall $a_n = a_{n-1} + 11a_{n-2} + 13a_{n-3}$.

Recall

$a_{n-1} \leq 5^{n-1}$  \hspace{1cm}  $a_{n-2} \leq 5^{n-2}$  \hspace{1cm}  $a_{n-3} \leq 5^{n-3}$.

\[
a_{n-1} + 11a_{n-2} + 13a_{n-3} \leq 5^{n-1} + 11 \times 5^{n-2} + 13 \times 5^{n-3}.
\]
Upper Bound

Recall $a_n = a_{n-1} + 11a_{n-2} + 13a_{n-3}$.

Recall

$$a_{n-1} \leq 5^{n-1} \quad a_{n-2} \leq 5^{n-2} \quad a_{n-3} \leq 5^{n-3}.$$  

$$a_{n-1} + 11a_{n-2} + 13a_{n-3} \leq 5^{n-1} + 11 \times 5^{n-2} + 13 \times 5^{n-3}.$$  

We WANT this to be $\leq 5^n$. Lets see:

$$5^{n-1} + 11 \times 5^{n-2} + 13 \times 5^{n-3} \leq 5^n$$
Upper Bound

Recall \( a_n = a_{n-1} + 11a_{n-2} + 13a_{n-3} \).

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\end{align*}
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\[
5^{n-1} + 11 \times 5^{n-2} + 13 \times 5^{n-3} \leq 5^n
\]

Divide by \( 5^{n-3} \) to get

\[
5^2 + 11 \times 5 + 13 \times 1 \leq 5^3
\]
Upper Bound

Recall \( a_n = a_{n-1} + 11a_{n-2} + 13a_{n-3} \).

Recall

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a_{n-1} \leq 5^{n-1} \quad a_{n-2} \leq 5^{n-2} \quad a_{n-3} \leq 5^{n-3}.
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5^2 + 11 \times 5 + 13 \times 1 \leq 5^3
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\[
25 + 55 + 13 \leq 125
\]
Upper Bound

Recall \( a_n = a_{n-1} + 11a_{n-2} + 13a_{n-3} \).

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5^2 + 11 \times 5 + 13 \times 1 \leq 5^3
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\[
25 + 55 + 13 \leq 125
\]

\[
63 \leq 125 \text{ TRUE!}
\]
How did I Know to take $5^n$?

(1) Fib: $f_n$ depends on $f_{n-1}$ and $f_{n-2}$. Fib is exponential.
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(2) $a_n$: $a_n$ depends on $a_{n-1}$, $a_{n-2}$, $a_{n-3}$: We **guess** exponential.
How did I Know to take $5^n$?

(1) Fib: $f_n$ depends on $f_{n-1}$ and $f_{n-2}$. Fib is exponential.
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(3) I did proof with $\alpha^n$; found least $\alpha \in \mathbb{N}$ that made proof work.

(4) Could use this to find an exact $\alpha$, but messy so we won't.

(5) This is called Constructive Induction. It's the topic of the next slide packet.
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