Strong Induction and Inequalities

Nice Recurrences

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$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ 8 & \text{if } n = 1 \\ a_{n-1} + 2a_{n-2} & \text{if } n \geq 2 \end{cases} \qquad (1)$$

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$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ 3 & \text{if } n = 2 \\ a_{n-1} + 11a_{n-2} + 13a_{n-3} & \text{if } n \geq 2 \end{cases} \tag{3}$$

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YES, and it only involves integers
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NO.
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YES, but it involves irrationals.

Rather than bother with an exact solution, we will prove an UPPER BOUND that IS nice.

Upper Bound

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Thm $(\forall n)[a_n \leq 5^n]$

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Base Case $a_0 = 1 \leq 5^0 = 1$ YES. Also $a_1 = 2 \leq 5^1 = 5$.

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$a_{n-1} \leq 5^{n-1}$,
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Finish on next slide.

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$a_{n-1} \le 5^{n-1} \qquad a_{n-2} \le 5^{n-2} \qquad a_{n-3} \le 5^{n-3}$.

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We WANT this to be $\le 5^n$. Lets see:

$$5^{n-1} + 11 \times 5^{n-2} + 13 \times 5^{n-3} \le 5^n$$

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$$25 + 55 + 13 \le 125$$

$$63 \le 125 \text{ TRUE!}$$

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(5) This is called **Constructive Induction**. It's the topic of the next slide packet.