Loaded Dice
Fair Dice Yield Unfair Sums

Fair Die:

\[ \text{Pr}(1) = \text{Pr}(2) = \text{Pr}(3) = \text{Pr}(4) = \text{Pr}(5) = \text{Pr}(6) = \frac{1}{6} \sim 0.167 \]

Roll TWO of them.

\[ \text{Pr}(\text{Sum} = 2) = \frac{1}{36} \text{ (This is Min Pr(Sum))} \]
\[ \text{Pr}(\text{Sum} = 7) = \frac{1}{6} \text{. (This is Max Pr(Sum))} \]
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Sums are Unfair!
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**Sums are Unfair!**

**How Unfair?:** \[ 1/6 - 1/36 \sim 0.139 \text{ unfair.} \]
What are Loaded Dice?

**Def** A Die is a 6-tuple \((p_1, p_2, p_3, p_4, p_5, p_6)\) such that \(0 \leq p_i \leq 1\) and \(\sum_{i=1}^{6} p_i = 1\).
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**Our Questions:**
1) Does there exist a pair of dice such that the sums all have equal probability \(1/11\)?
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1) There exists a way to load dice so that all sums are prob \(1/11\).
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1) Does there exist a pair of dice such that the sums all have equal probability \(1/11\)?

What Do You Think?

VOTE

1) There exists a way to load dice so that all sums are prob \(\frac{1}{11}\).
2) There is no way to load dice so that all the sums are prob \(\frac{1}{11}\).

No such dice can exist!
Polynomials are our Friends!

Assume that are dice that yield fair sums. Let \((p_1, \ldots, p_6)\) and \((q_1, \ldots, q_6)\) be those dice.

KEY:

\[(p_1 x + p_2 x^2 + \cdots + p_6 x^6)(q_1 x + q_2 x^2 + \cdots + q_6 x^6)\]

Coefficient of \(x^5\) is

\[p_1 q_4 + p_2 q_3 + p_3 q_2 + p_4 q_1 = \text{Prob}(\text{sum} = 5)\]

Coefficient of \(x^i\) is \(\text{Prob}(\text{sum} = i)\).
Fair Sums- NOT!

Let \((p_1, \ldots, p_6)\) and \((q_1, \ldots, q_6)\) be dice. Assume they yield FAIR SUMS, all sums have prob \(1/11\). Then

\[
(p_1 x + \cdots + p_6 x^6)(q_1 x + \cdots + q_6 x^6) = (1/11)(x^2 + x^3 + \cdots + x^{12})
\]

So

\[
(p_1 + \cdots + p_6 x^5)(q_1 + \cdots + q_6 x^5) = (1/11)(1 + x + x^2 + \cdots + x^{10})
\]
**Two Polynomials**

**Recap** If \((p_1, \ldots, p_6)\) and \((q_1, \ldots, q_6)\) are two loaded dice that yield fair sums then:

\[
(p_1 + \cdots + p_6 x^5)(q_1 + \cdots + q_6 x^5) = \left(\frac{1}{11}\right)(1 + x + x^2 + \cdots + x^{10})
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Two Polynomials

Recap If \((p_1, \ldots, p_6)\) and \((q_1, \ldots, q_6)\) are two loaded dice that yield fair sums then:

\[(p_1 + \cdots + p_6 x^5)(q_1 + \cdots + q_6 x^5) = \frac{1}{11}(1 + x + x^2 + \cdots + x^{10})\]

\[(1 + x + \cdots + x^{10}) = \frac{x^{11} - 1}{x - 1} \quad \text{hence}\]
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hence

\[
(p_1 + \cdots + p_6 x^5)(q_1 + \cdots + q_6 x^5)(x - 1) = x^{11} - 1
\]
Real Roots of Left Polynomials

\[(p_1 + \cdots + p_6x^5)(q_1 + \cdots + q_6x^5)(x - 1) = x^{11} - 1\]
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\(q_1 + \cdots + q_6x^5\): odd degree, real coefficients, so has \(\geq 1\) real root.
\((x - 1)\) has 1 real root.

Upshot: The Left poly has \(\geq 3\) real roots.
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\[(x - 1) \text{ has 1 real root.}\]

**Upshot** The Left poly has \( \geq 3 \) real roots.
Real Roots of Right Polynomials (II)

\[(p_1 + \cdots + p_6 x^5)(q_1 + \cdots + q_6 x^5)(x - 1) = x^{11} - 1\]

Recall Upshot
The Left poly has $\geq 3$ real roots.

Let's look at the roots of the right poly:
\[x^{11} - 1 = 0\]
\[x^{11} = 1\]

All roots on complex unit circle. Hence $\leq 2$ real roots.

Upshot
The Right poly has $\leq 2$ real roots.

Final Upshot
The left and right poly DIFFER on the number of real roots, so they cannot be the same. Contradiction!
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