Logic Problems

250H

Give a Propositional Formula on four variables that has exactly three satisfying assignments. Give the satisfying assignments.

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$$(x_1 \wedge x_2 \wedge x_3 \wedge x_4) \vee (\neg(x_1) \wedge x_2 \wedge x_3 \wedge x_4) \vee (\neg(x_1) \wedge \neg(x_2) \wedge x_3 \wedge x_4)$$

The only satisfying assignments are (T, T, T, T) and (F, T, T, T) and (F, F, T, T).

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The only way $p \Rightarrow (q \Rightarrow r)$ is F is if p is T and $(q \Rightarrow r)$ is F. The latter can only happen if q is T and r is F. Hence the only way $p \Rightarrow (q \Rightarrow r)$ is F is if p is T, q is T, and r is F.

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The only way $(p \Rightarrow q) \Rightarrow r$ is F is if $(p \Rightarrow q)$ is T and r is F. $(p \Rightarrow q)$ is T when (p, q) is either (T, T), (F, T) or (F, F). Hence $(p \Rightarrow q) \Rightarrow r$ is F when (p, q, r) is either (T, T, F), (F, T, F), (F, F, F).

p	q	r	$p \Rightarrow (q \Rightarrow r)$	$(p \Rightarrow q) \Rightarrow r$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	F
F	F	T	T	T
F	F	F	T	F

p	q	r	$p \Rightarrow (q \Rightarrow r)$	$(p \Rightarrow q) \Rightarrow r$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	F
F	F	T	T	T
F	F	F	T	F

NOT equiv: they differ on the row (F,T,F).

Show that, for all $n \geq 1$, there exists a formula that is satisfied by exactly n satisfying assignments. Give the satisfying assignments.

Let $\phi(i, n)$ be the Boolean formula on $x_1, \ldots, x_n$ where $x_1, \ldots, x_i$ are NEGATED but the rest are not. For examples

$\phi(0, n)$ is $(x_1 \wedge x_2 \wedge \cdots \wedge x_n)$

$\phi(3, n)$ is $(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge x_5 \wedge \cdots \wedge x_n)$

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$\phi(3, n)$ is $(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge x_5 \wedge \cdots \wedge x_n)$

Our formula is

$$\phi(0, n) \vee \phi(1, n) \vee \cdots \vee \phi(n-1, n)$$

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The satisfying assignments are

$(T, \ldots, T)$ (n T's)

$(F, T, \ldots, T)$ (1 F and then $n-1$ T's)

$(F, F, T, \ldots, T)$ (2 F's and then $n-2$ T's)

$\vdots$

$(F, F, F, \ldots, F, T)$ ($n-1$ F's and then 1 T)

There are other options. Perhaps more pleasing would be

$$\phi(1,n) \vee \cdots \vee \phi(n,n).$$

Consider the statement:

$$\text{for all } n \; [\neg(p_1 \wedge \cdots \wedge p_n) \equiv \neg p_1 \vee \cdots \vee \neg p_n]$$

Prove it. Note that you cannot use Truth Table since we want it for all n.

First consider: for which rows will

$$\neg(p_1 \wedge \cdots \wedge p_n)$$

be FALSE? Only if $p_1 \wedge \cdots \wedge p_n$ is TRUE. Since its a $\wedge$ that only happens if ALL of the p_i's are TRUE. So we have:

$\neg(p_1 \wedge \cdots \wedge p_n)$ FALSE if and only if ALL p_i's are TRUE.

Second consider: for which rows will

$$\neg p_1 \vee \cdots \vee \neg p_n$$

be FALSE? Since its a $\vee$ this only happens if each literal is FALSE. So we need, for all i, $\neg p_i$ is FALSE. So we need, for all i, p_i is TRUE. So we have:

$$\neg p_1 \vee \cdots \vee \neg p_n$$

FALSE if and only if all p_i's are TRUE.

Since the two statements are FALSE for all the same rows, they are also TRUE for all the same rows. Hence they are equivalent.

For each of the following statements write the negation without using any negations signs.

(a) $x \leq 4$

(b) $1 < x < 2$

(c) $x_1 < x_2 < \cdots < x_n$

(d) $x \leq 5$ OR $x \geq 10$

(e) $x \leq 5$ AND $x \geq 10$

a) $x > 4$.

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b) $x \leq 1$ OR $x \geq 2$.

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b) $x \leq 1$ OR $x \geq 2$.

c) This can be rewritten as

$$\bigwedge_{1 \leq i \leq n-1} x_i < x_{i+1}$$

So its negation is

$$\bigvee_{1 \leq i \leq n-1} x_{i+1} \leq x_i$$

d) $1 < x < 10$. Also a fine answer: $1 < x$ AND $x < 10$.

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e) $x > 5$ OR $x < 10$. This one is a bit odd since the original statement can never happen. But the negation of something that can't happen is something that can happen.

Simplify the following formula so that its of the form QUANTIFIER QUANTIFIER then stuff. In other words, there is no negation on the outside or between the quantifiers.

$$\neg(\forall x)(\exists y)[R(x, y) \wedge \neg S(x, y)].$$

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$$\neg(\forall x)(\exists y)[R(x, y) \wedge \neg S(x, y)].$$

$$(\exists x)(\forall y)\neg[R(x, y) \wedge \neg S(x, y)]$$

$$(\exists x)(\forall y)[\neg R(x, y) \vee S(x, y)]$$