Reciprocal Theorems
THE Reciprocal Theorem

Thm \((\forall n \geq 3)(\exists d_1 < \cdots < d_n)\) such that

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We will proof this theorem an infinite number of ways.
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All of them will be by induction.
Base Case for All of the Proofs

We will usually only need the $n = 3$ base case:
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We may sometimes need the $n = 4$ base case:
$\frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{24} = 1$. 
Proof One. This was on Midterm Two
IH $n \geq 3$. There exists $d_1 < \cdots < d_n$ such that

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$$1 = \frac{1}{d_1} + \cdots + \frac{1}{d_n} = \frac{1}{d_1} + \cdots + \frac{1}{d_{n-1}} + \frac{1}{d_n+1} + \frac{1}{d_n(d_n+1)}.$$
Proof Two. Bigger Base Case and

\[ P(n) \rightarrow P(n + 2) \]
An Induction Scheme

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\[(\forall n \geq 3)[P(n) \rightarrow P(n + 2)].\]
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This Works! From the above you can construct a proof of \(P(n)\) for any \(n \geq 3\).
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For the case at hand we already did the \(n = 3\) and \(n = 4\) base case.
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We use that $\frac{1}{d_n} = \frac{1}{2d_n} + \frac{1}{3d_n} + \frac{1}{6d_n}$.
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1 = \frac{1}{d_1} + \cdots + \frac{1}{d_n} = \frac{1}{d_1} + \cdots + \frac{1}{d_{n-1}} + \frac{1}{2d_n} + \frac{1}{3d_n} + \frac{1}{6d_n}.
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Generalization of Proof Two

Proof 2 used

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Can we use any way to write 1 as a sum of reciprocals? Our next proof does this and make some other points of interest.
Proof Three. Load the IH
Key Equation

Note that

\[ 1 = \frac{1}{3/2} + \frac{1}{3}. \]

Can we use this? Let's try to use it manually.
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Working Things Out By Hand

\[ 1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} \]

Can we keep doing this?

Yes.

Can we make this process into a rigorous proof?

Discuss

It works so long as the last number is \( \equiv 0 \pmod{2} \).
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\[ \frac{1}{d} = \frac{1}{3d/2} + \frac{1}{3d} \]

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Proof a Harder Theorem

Convention $\equiv \equiv \ (\text{mod } 2)$.
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**Convention** \( \equiv \) means \( \equiv \pmod{2} \).

**Thm** \( (\forall n \geq 3)(\exists d_1 < \cdots < d_n, d_n \equiv 0) \) such that
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**Loading the IH** Proving a harder theorem so that the IH is stronger.
IH and IS

IB  \( d = 3 \).  \( 1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}, \ 6 \equiv 0. \)
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We use that $\frac{1}{d_n} = \frac{1}{3d_n/2} + \frac{1}{3d_n}$. Since $d_n \equiv 0$, $3d_n/2 \in \mathbb{N}$. 

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$$1 = \frac{1}{d_1} + \cdots + \frac{1}{d_n} = \frac{1}{d_1} + \cdots + \frac{1}{d_{n-1}} + \frac{1}{3d_n/2} + \frac{1}{3d_n}.$$

Also NEED that the last number is $\equiv 0$. It is since $3d_n \equiv d_n \equiv 0$. 
Proof Four. A Different Approach
IH $n \geq 3$. There exists $d_1 < \cdots < d_n$ such that

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