BILL AND EMILY RECORD LECTURE!!!!

UNTIMED PART OF FINAL IS MONDAY May 10 9:00AM. NO DEAD CAT

FINAL IS MONDAY May 17 8:00PM-10:15PM

FILL OUT COURSE EVALS for ALL YOUR COURSES!!!

Solving Recurrences

Solving Recurrences: Fib

Recall Fib Formula

Recall the Fib Sequence:
$a_0 = 0$
$a_1 = 1$
$(\forall n \geq 2)[a_n = a_{n-1} + a_{n-2}]$.

Recall Fib Formula

Recall the Fib Sequence:
$a_0 = 0$
$a_1 = 1$
$(\forall n \geq 2)[a_n = a_{n-1} + a_{n-2}]$.

Recall the formula we gave for it.

Recall Fib Formula

Recall the Fib Sequence:
$a_0 = 0$
$a_1 = 1$
$(\forall n \geq 2)[a_n = a_{n-1} + a_{n-2}]$.

Recall the formula we gave for it.
Let $\alpha_1 = \frac{1+\sqrt{5}}{2}$ and $\alpha_2 = \frac{1-\sqrt{5}}{2}$.

Recall Fib Formula

Recall the Fib Sequence:
$a_0 = 0$
$a_1 = 1$
$(\forall n \geq 2)[a_n = a_{n-1} + a_{n-2}]$.
Recall the formula we gave for it.
Let $\alpha_1 = \frac{1+\sqrt{5}}{2}$ and $\alpha_2 = \frac{1-\sqrt{5}}{2}$.
Then

$$a_n = \frac{\alpha_1^n - \alpha_2^n}{\sqrt{5}}.$$

Recall Fib Formula

Recall the Fib Sequence:
$a_0 = 0$
$a_1 = 1$
$(\forall n \geq 2)[a_n = a_{n-1} + a_{n-2}]$.
Recall the formula we gave for it.
Let $\alpha_1 = \frac{1+\sqrt{5}}{2}$ and $\alpha_2 = \frac{1-\sqrt{5}}{2}$.
Then

$$a_n = \frac{\alpha_1^n - \alpha_2^n}{\sqrt{5}}.$$

We could **prove** the formula by a painful algebraic induction.

Recall Fib Formula

Recall the Fib Sequence:
$a_0 = 0$
$a_1 = 1$
$(\forall n \geq 2)[a_n = a_{n-1} + a_{n-2}]$.
Recall the formula we gave for it.
Let $\alpha_1 = \frac{1+\sqrt{5}}{2}$ and $\alpha_2 = \frac{1-\sqrt{5}}{2}$.
Then

$$a_n = \frac{\alpha_1^n - \alpha_2^n}{\sqrt{5}}.$$

We could **prove** the formula by a painful algebraic induction.

Better idea Lets Derive it.

Ignore The Base Case for a While

Plan For now find **all** solutions to

$$a_n = a_{n-1} + a_{n-2}$$

Ignore The Base Case for a While

Plan For now find **all** solutions to

$$a_n = a_{n-1} + a_{n-2}$$

We will combine them and modify them to fit base case.

Solutions are Additive

Assume $f(n)$ and $g(n)$ both satisfy

$$a_n = a_{n-1} + a_{n-2}.$$

Solutions are Additive

Assume $f(n)$ and $g(n)$ both satisfy

$$a_n = a_{n-1} + a_{n-2}.$$

Then for any constants c, d, $cf(n) + dg(n)$ satisfies

$$a_n = a_{n-1} + a_{n-2}.$$

Solutions are Additive

Assume $f(n)$ and $g(n)$ both satisfy

$$a_n = a_{n-1} + a_{n-2}.$$

Then for any constants c, d, $cf(n) + dg(n)$ satisfies

$$a_n = a_{n-1} + a_{n-2}.$$

This is just algebra which we will skip.

Solutions are Additive

Assume $f(n)$ and $g(n)$ both satisfy

$$a_n = a_{n-1} + a_{n-2}.$$

Then for any constants c, d, $cf(n) + dg(n)$ satisfies

$$a_n = a_{n-1} + a_{n-2}.$$

This is just algebra which we will skip.

Upshot The set of solutions to $a_n = a_{n-1} + a_{n-2}$ is closed under addition and scalar multiplication (its a vector space).

Guess a Solution to Just the Recurrence Part

We will guess that α^n satisfies the recurrence, so

Guess a Solution to Just the Recurrence Part

We will guess that α^n satisfies the recurrence, so

$$\alpha^n = \alpha^{n-1} + \alpha^{n-2}$$

Guess a Solution to Just the Recurrence Part

We will guess that α^n satisfies the recurrence, so

$$\alpha^n = \alpha^{n-1} + \alpha^{n-2}$$

Factor out α^{n-2} to get

Guess a Solution to Just the Recurrence Part

We will guess that α^n satisfies the recurrence, so

$$\alpha^n = \alpha^{n-1} + \alpha^{n-2}$$

Factor out α^{n-2} to get

$$\alpha^2 = \alpha + 1$$

$$\alpha^2 - \alpha - 1 = 0$$

Guess a Solution to Just the Recurrence Part

We will guess that α^n satisfies the recurrence, so

$$\alpha^n = \alpha^{n-1} + \alpha^{n-2}$$

Factor out α^{n-2} to get

$$\alpha^2 = \alpha + 1$$

$$\alpha^2 - \alpha - 1 = 0$$

Roots are $\alpha_1 = \frac{1+\sqrt{5}}{5}$ and $\alpha_2 = \frac{1-\sqrt{5}}{5}$.

Guess a Solution to Just the Recurrence Part

We will guess that α^n satisfies the recurrence, so

$$\alpha^n = \alpha^{n-1} + \alpha^{n-2}$$

Factor out α^{n-2} to get

$$\alpha^2 = \alpha + 1$$

$$\alpha^2 - \alpha - 1 = 0$$

Roots are $\alpha_1 = \frac{1+\sqrt{5}}{5}$ and $\alpha_2 = \frac{1-\sqrt{5}}{5}$.
α_1^n and α_2^n both satisfy the recurrence.

Guess a Solution to Just the Recurrence Part

We will guess that α^n satisfies the recurrence, so

$$\alpha^n = \alpha^{n-1} + \alpha^{n-2}$$

Factor out α^{n-2} to get

$$\alpha^2 = \alpha + 1$$

$$\alpha^2 - \alpha - 1 = 0$$

Roots are $\alpha_1 = \frac{1+\sqrt{5}}{5}$ and $\alpha_2 = \frac{1-\sqrt{5}}{5}$.

α_1^n and α_2^n both satisfy the recurrence.

Upshot For any constants c, d $c\alpha_1^n + d\alpha_2^n$ satisfy the recurrence.

Its all about the Base, bout the Base...

For any constants c, d $c\alpha_1^n + d\alpha_2^n$ satisfy the recurrence.

Its all about the Base, bout the Base...

For any constants c, d $c\alpha_1^n + d\alpha_2^n$ satisfy the recurrence.

We want to pick c, d so that the base case is satisfied.

Its all about the Base, bout the Base...

For any constants c, d $c\alpha_1^n + d\alpha_2^n$ satisfy the recurrence.

We want to pick c, d so that the base case is satisfied.

$a_0 = 0$: $c\alpha_1^0 + d\alpha_2^0 = 0$

$$c + d = 1.$$

Its all about the Base, bout the Base...

For any constants c, d $c\alpha_1^n + d\alpha_2^n$ satisfy the recurrence.

We want to pick c, d so that the base case is satisfied.

$a_0 = 0$: $c\alpha_1^0 + d\alpha_2^0 = 0$

$$c + d = 1.$$

$a_1 = 1$. $c\alpha_1^1 + d\alpha_2^1 = 0$

$$c\alpha_1 + d\alpha_2 = 1$$

Its all about the Base, bout the Base. . .

For any constants c, d $c\alpha_1^n + d\alpha_2^n$ satisfy the recurrence.

We want to pick c, d so that the base case is satisfied.

$a_0 = 0$: $c\alpha_1^0 + d\alpha_2^0 = 0$

$$c + d = 1.$$

$a_1 = 1$. $c\alpha_1^1 + d\alpha_2^1 = 0$

$$c\alpha_1 + d\alpha_2 = 1$$

Two linear equations in two variables can be solved:

Its all about the Base, bout the Base...

For any constants c, d $c\alpha_1^n + d\alpha_2^n$ satisfy the recurrence.

We want to pick c, d so that the base case is satisfied.

$a_0 = 0$: $c\alpha_1^0 + d\alpha_2^0 = 0$

$$c + d = 1.$$

$a_1 = 1$. $c\alpha_1^1 + d\alpha_2^1 = 0$

$$c\alpha_1 + d\alpha_2 = 1$$

Two linear equations in two variables can be solved:
$c = \frac{1}{\sqrt{5}}$, $d = -\frac{1}{\sqrt{5}}$,

Its all about the Base, bout the Base...

For any constants c, d $c\alpha_1^n + d\alpha_2^n$ satisfy the recurrence.

We want to pick c, d so that the base case is satisfied.

$a_0 = 0$: $c\alpha_1^0 + d\alpha_2^0 = 0$

$$c + d = 1.$$

$a_1 = 1$. $c\alpha_1^1 + d\alpha_2^1 = 0$

$$c\alpha_1 + d\alpha_2 = 1$$

Two linear equations in two variables can be solved:
$c = \frac{1}{\sqrt{5}}$, $d = -\frac{1}{\sqrt{5}}$,
Upshot The recurrence is solved by

$$a_n = \frac{\alpha_1^n - \alpha_2^n}{\sqrt{5}}.$$

Solving Recurrences: Distinct Roots Case

General Problem

Given $a_0, a_1, \ldots, a_{k-1}$ and

General Problem

Given $a_0, a_1, \ldots, a_{k-1}$ and

$(\forall n \geq k)[a_n = b_{n-1}a_{n-1} + \cdots + b_{n-k}a_{n-k}]$

General Problem

Given $a_0, a_1, \ldots, a_{k-1}$ and

$(\forall n \geq k)[a_n = b_{n-1}a_{n-1} + \cdots + b_{n-k}a_{n-k}]$

Find a closed formula for a_n.

Why Do We Care?

Why do we care about solving recurrences of the form
Given $a_0, a_1, \ldots, a_{k-1}$ and

Why Do We Care?

Why do we care about solving recurrences of the form
Given $a_0, a_1, \ldots, a_{k-1}$ and
$(\forall n \geq k)[a_n = b_{n-1}a_{n-1} + \cdots + b_{n-k}a_{n-k}]$

Why Do We Care?

Why do we care about solving recurrences of the form
Given $a_0, a_1, \ldots, a_{k-1}$ and
$(\forall n \ge k)[a_n = b_{n-1}a_{n-1} + \cdots + b_{n-k}a_{n-k}]$
(1) They come up in problems in combinatorics.

Why Do We Care?

Why do we care about solving recurrences of the form
Given $a_0, a_1, \ldots, a_{k-1}$ and
$(\forall n \geq k)[a_n = b_{n-1}a_{n-1} + \cdots + b_{n-k}a_{n-k}]$
(1) They come up in problems in combinatorics.
(2) They model some real world phenomena like Population Growth or the spread of an infection.

Why Do We Care?

Why do we care about solving recurrences of the form
Given $a_0, a_1, \ldots, a_{k-1}$ and
$(\forall n \ge k)[a_n = b_{n-1}a_{n-1} + \cdots + b_{n-k}a_{n-k}]$
(1) They come up in problems in combinatorics.
(2) They model some real world phenomena like Population Growth or the spread of an infection.
(3) To solve Differential Equations sometimes they are made discrete and become difference equations.

Why Do We Care?

Why do we care about solving recurrences of the form
Given $a_0, a_1, \ldots, a_{k-1}$ and
$(\forall n \geq k)[a_n = b_{n-1}a_{n-1} + \cdots + b_{n-k}a_{n-k}]$
(1) They come up in problems in combinatorics.
(2) They model some real world phenomena like Population Growth or the spread of an infection.
(3) To solve Differential Equations sometimes they are made discrete and become difference equations.
(4) Note: In CMSC 351 you will look at equations like

$$a_n = 2a_{n/2} + n$$

Which are used to analyze algorithms. That is NOT todays topic.

Step One: Ignore the Base

Guess that α^n satisfies

$$a_n = b_{n-1}a_{n-1} + \cdots + b_{n-k}a_{n-k}$$

Step One: Ignore the Base

Guess that α^n satisfies

$$a_n = b_{n-1}a_{n-1} + \cdots + b_{n-k}a_{n-k}$$

$$\alpha^n = b_{n-1}\alpha^{n-1} + \cdots + b_{n-k}\alpha^{n-k}$$

Step One: Ignore the Base

Guess that α^n satisfies

$$a_n = b_{n-1}a_{n-1} + \cdots + b_{n-k}a_{n-k}$$

$$\alpha^n = b_{n-1}\alpha^{n-1} + \cdots + b_{n-k}\alpha^{n-k}$$

$$\alpha^k - b_{n-1}\alpha^{k-1} - \cdots - b_{n-k+1}\alpha - b_{n-k} = 0.$$

Step One: Ignore the Base

Guess that α^n satisfies

$$a_n = b_{n-1}a_{n-1} + \cdots + b_{n-k}a_{n-k}$$

$$\alpha^n = b_{n-1}\alpha^{n-1} + \cdots + b_{n-k}\alpha^{n-k}$$

$$\alpha^k - b_{n-1}\alpha^{k-1} - \cdots - b_{n-k+1}\alpha - b_{n-k} = 0.$$

Let $\alpha_1, \ldots, \alpha_k$ be the roots.

Step One: Ignore the Base

Guess that α^n satisfies

$$a_n = b_{n-1}a_{n-1} + \cdots + b_{n-k}a_{n-k}$$

$$\alpha^n = b_{n-1}\alpha^{n-1} + \cdots + b_{n-k}\alpha^{n-k}$$

$$\alpha^k - b_{n-1}\alpha^{k-1} - \cdots - b_{n-k+1}\alpha - b_{n-k} = 0.$$

Let $\alpha_1, \ldots, \alpha_k$ be the roots.

We assume they are distinct. (Non-distinct case later.)

Step One: Ignore the Base

Guess that α^n satisfies

$$a_n = b_{n-1}a_{n-1} + \cdots + b_{n-k}a_{n-k}$$

$$\alpha^n = b_{n-1}\alpha^{n-1} + \cdots + b_{n-k}\alpha^{n-k}$$

$$\alpha^k - b_{n-1}\alpha^{k-1} - \cdots - b_{n-k+1}\alpha - b_{n-k} = 0.$$

Let $\alpha_1, \ldots, \alpha_k$ be the roots.

We assume they are distinct. (Non-distinct case later.)

Upshot For any constants $c_1, \ldots, c_k$

$$c_1\alpha_1^n + \cdots + c_k\alpha_k^n$$

is a solution to the recurrence.

Its all about the Base, bout the Base...

For $0 \leq L \leq k-1$:
Use a_L:

$$c_1\alpha_1^L + \cdots + c_k\alpha_k^L = a_L$$

Its all about the Base, bout the Base...

For $0 \le L \le k-1$:
Use a_L:

$$c_1\alpha_1^L + \cdots + c_k\alpha_k^L = a_L$$

This gives k linear equations in k variables.

Its all about the Base, bout the Base...

For $0 \leq L \leq k-1$:
Use a_L:

$$c_1\alpha_1^L + \cdots + c_k\alpha_k^L = a_L$$

This gives k linear equations in k variables.

That can be solved.

Its all about the Base, bout the Base...

For $0 \leq L \leq k-1$:
Use a_L:

$$c_1\alpha_1^L + \cdots + c_k\alpha_k^L = a_L$$

This gives k linear equations in k variables.

That can be solved.

Then you have the closed form solution.

Solving Recurrences: The Non-Distinct Roots Case

An Example

Recall the Bill Sequence:

$a_0 = 0$

$a_1 = 1$

$a_2 = 2$

$(\forall n \geq 2)[a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}]$.

Ignore The Base Case for a While

Plan For now find **all** solutions to
$(\forall n \geq 2)[a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}]$.

Ignore The Base Case for a While

Plan For now find **all** solutions to
$(\forall n \ge 2)[a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}]$.
We will combine them and modify them to fit base case.

Ignore The Base Case for a While

Plan For now find **all** solutions to
$(\forall n \geq 2)[a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}]$.
We will combine them and modify them to fit base case.
Recall The set of solutions to is closed under addition and scalar multiplication (its a vector space).

Guess a Solution to Just the Recurrence Part

We will guess that α^n satisfies the recurrence, so

Guess a Solution to Just the Recurrence Part

We will guess that α^n satisfies the recurrence, so

$$\alpha^n = 7\alpha^{n-1} - 16\alpha^{n-2} + 12\alpha^{n-3}$$

Guess a Solution to Just the Recurrence Part

We will guess that α^n satisfies the recurrence, so

$$\alpha^n = 7\alpha^{n-1} - 16\alpha^{n-2} + 12\alpha^{n-3}$$

Factor out α^{n-3} to get

Guess a Solution to Just the Recurrence Part

We will guess that α^n satisfies the recurrence, so

$$\alpha^n = 7\alpha^{n-1} - 16\alpha^{n-2} + 12\alpha^{n-3}$$

Factor out α^{n-3} to get

$$\alpha^3 = 7\alpha^2 - 16\alpha + 12$$

Guess a Solution to Just the Recurrence Part

We will guess that α^n satisfies the recurrence, so

$$\alpha^n = 7\alpha^{n-1} - 16\alpha^{n-2} + 12\alpha^{n-3}$$

Factor out α^{n-3} to get

$$\alpha^3 = 7\alpha^2 - 16\alpha + 12$$

$$\alpha^3 - 7\alpha^2 + 16\alpha - 12 = 0$$

Guess a Solution to Just the Recurrence Part

We will guess that α^n satisfies the recurrence, so

$$\alpha^n = 7\alpha^{n-1} - 16\alpha^{n-2} + 12\alpha^{n-3}$$

Factor out α^{n-3} to get

$$\alpha^3 = 7\alpha^2 - 16\alpha + 12$$

$$\alpha^3 - 7\alpha^2 + 16\alpha - 12 = 0$$

$$(\alpha - 2)^2(\alpha - 3)$$

Guess a Solution to Just the Recurrence Part

We will guess that α^n satisfies the recurrence, so

$$\alpha^n = 7\alpha^{n-1} - 16\alpha^{n-2} + 12\alpha^{n-3}$$

Factor out α^{n-3} to get

$$\alpha^3 = 7\alpha^2 - 16\alpha + 12$$

$$\alpha^3 - 7\alpha^2 + 16\alpha - 12 = 0$$

$$(\alpha - 2)^2(\alpha - 3)$$

Roots are $\alpha_1 = 2$, $\alpha_2 = 2$, $\alpha_3 = 3$.

Guess a Solution to Just the Recurrence Part

We will guess that α^n satisfies the recurrence, so

$$\alpha^n = 7\alpha^{n-1} - 16\alpha^{n-2} + 12\alpha^{n-3}$$

Factor out α^{n-3} to get

$$\alpha^3 = 7\alpha^2 - 16\alpha + 12$$

$$\alpha^3 - 7\alpha^2 + 16\alpha - 12 = 0$$

$$(\alpha - 2)^2(\alpha - 3)$$

Roots are $\alpha_1 = 2$, $\alpha_2 = 2$, $\alpha_3 = 3$.

α_1^n and α_2^n and α_3^n all satisfy the recurrence.

Guess a Solution to Just the Recurrence Part

We will guess that α^n satisfies the recurrence, so

$$\alpha^n = 7\alpha^{n-1} - 16\alpha^{n-2} + 12\alpha^{n-3}$$

Factor out α^{n-3} to get

$$\alpha^3 = 7\alpha^2 - 16\alpha + 12$$

$$\alpha^3 - 7\alpha^2 + 16\alpha - 12 = 0$$

$$(\alpha - 2)^2(\alpha - 3)$$

Roots are $\alpha_1 = 2$, $\alpha_2 = 2$, $\alpha_3 = 3$.

α_1^n and α_2^n and α_3^n all satisfy the recurrence.

But we need **three** solutions to make all of this work.

This is So Crazy it Might Just Work

Lets see if $n2^n$ is a solution to just the recurrence.

This is So Crazy it Might Just Work

Lets see if $n2^n$ is a solution to just the recurrence.

$$a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}$$

This is So Crazy it Might Just Work

Lets see if $n2^n$ is a solution to just the recurrence.

$$a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}$$

We hope:

This is So Crazy it Might Just Work

Lets see if $n2^n$ is a solution to just the recurrence.

$$a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}$$

We hope:

$$n2^n = 7 \times (n-1)2^{n-1} - 16(n-2)2^{n-2} + 12(n-3)2^{n-3}$$

This is So Crazy it Might Just Work

Lets see if $n2^n$ is a solution to just the recurrence.

$$a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}$$

We hope:

$$n2^n = 7 \times (n-1)2^{n-1} - 16(n-2)2^{n-2} + 12(n-3)2^{n-3}$$

$$n2^3 = 7 \times (n-1)2^2 - 16(n-2)2 + 12(n-3)$$

This is So Crazy it Might Just Work

Lets see if $n2^n$ is a solution to just the recurrence.

$$a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}$$

We hope:

$$n2^n = 7 \times (n-1)2^{n-1} - 16(n-2)2^{n-2} + 12(n-3)2^{n-3}$$

$$n2^3 = 7 \times (n-1)2^2 - 16(n-2)2 + 12(n-3)$$

$$8n = 28(n-1) - 32(n-2) + 12(n-3)$$

This is So Crazy it Might Just Work

Lets see if $n2^n$ is a solution to just the recurrence.

$$a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}$$

We hope:

$$n2^n = 7 \times (n-1)2^{n-1} - 16(n-2)2^{n-2} + 12(n-3)2^{n-3}$$

$$n2^3 = 7 \times (n-1)2^2 - 16(n-2)2 + 12(n-3)$$

$$8n = 28(n-1) - 32(n-2) + 12(n-3)$$

$$8n = 28n - 28 - 32n + 64 + 12n - 36$$

This is So Crazy it Might Just Work

Lets see if $n2^n$ is a solution to just the recurrence.

$$a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}$$

We hope:

$$n2^n = 7 \times (n-1)2^{n-1} - 16(n-2)2^{n-2} + 12(n-3)2^{n-3}$$

$$n2^3 = 7 \times (n-1)2^2 - 16(n-2)2 + 12(n-3)$$

$$8n = 28(n-1) - 32(n-2) + 12(n-3)$$

$$8n = 28n - 28 - 32n + 64 + 12n - 36$$

$$8n = (28 - 32 + 12)n + (64 - 28 - 36) = 8n$$

This is So Crazy it Might Just Work

Lets see if $n2^n$ is a solution to just the recurrence.

$$a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}$$

We hope:

$$n2^n = 7 \times (n-1)2^{n-1} - 16(n-2)2^{n-2} + 12(n-3)2^{n-3}$$

$$n2^3 = 7 \times (n-1)2^2 - 16(n-2)2 + 12(n-3)$$

$$8n = 28(n-1) - 32(n-2) + 12(n-3)$$

$$8n = 28n - 28 - 32n + 64 + 12n - 36$$

$$8n = (28 - 32 + 12)n + (64 - 28 - 36) = 8n$$

OH, that worked!

General Problem

Given $a_0, a_1, \ldots, a_{k-1}$ and

General Problem

Given $a_0, a_1, \ldots, a_{k-1}$ and

$(\forall n \ge k)[a_n = b_{n-1}a_{n-1} + \cdots + b_{n-k}a_{n-k}]$

General Problem

Given $a_0, a_1, \ldots, a_{k-1}$ and

$(\forall n \geq k)[a_n = b_{n-1}a_{n-1} + \cdots + b_{n-k}a_{n-k}]$

Find a closed formula for a_n.

Step One: Ignore the Base

Guess that α^n satisfies

$$a_n = b_{n-1}a_{n-1} + \cdots + b_{n-k}a_{n-k}$$

Step One: Ignore the Base

Guess that α^n satisfies

$$a_n = b_{n-1}a_{n-1} + \cdots + b_{n-k}a_{n-k}$$

$$\alpha^n = b_{n-1}\alpha^{n-1} + \cdots + b_{n-k}\alpha^{n-k}$$

$$\alpha^k - b_{n-1}\alpha^{k-1} - \cdots - b_{n-k+1}\alpha - b_{n-k} = 0.$$

Step One: Ignore the Base

Guess that α^n satisfies

$$a_n = b_{n-1}a_{n-1} + \cdots + b_{n-k}a_{n-k}$$

$$\alpha^n = b_{n-1}\alpha^{n-1} + \cdots + b_{n-k}\alpha^{n-k}$$

$$\alpha^k - b_{n-1}\alpha^{k-1} - \cdots - b_{n-k+1}\alpha - b_{n-k} = 0.$$

Let $\alpha_1, \ldots, \alpha_k$ be the roots.

Step One: Ignore the Base

Guess that α^n satisfies

$$a_n = b_{n-1}a_{n-1} + \cdots + b_{n-k}a_{n-k}$$

$$\alpha^n = b_{n-1}\alpha^{n-1} + \cdots + b_{n-k}\alpha^{n-k}$$

$$\alpha^k - b_{n-1}\alpha^{k-1} - \cdots - b_{n-k+1}\alpha - b_{n-k} = 0.$$

Let $\alpha_1, \ldots, \alpha_k$ be the roots.

Might not be distinct.

Step One: Ignore the Base

Guess that α^n satisfies

$$a_n = b_{n-1}a_{n-1} + \cdots + b_{n-k}a_{n-k}$$

$$\alpha^n = b_{n-1}\alpha^{n-1} + \cdots + b_{n-k}\alpha^{n-k}$$

$$\alpha^k - b_{n-1}\alpha^{k-1} - \cdots - b_{n-k+1}\alpha - b_{n-k} = 0.$$

Let $\alpha_1, \ldots, \alpha_k$ be the roots.

Might not be distinct.

If α_i appears L times then
α_i^n, $n\alpha_i^n$, $n^2\alpha_i^n$, …, $n^{L-1}\alpha_i^n$ are solutions.

Step One: Ignore the Base

Guess that α^n satisfies

$$a_n = b_{n-1}a_{n-1} + \cdots + b_{n-k}a_{n-k}$$

$$\alpha^n = b_{n-1}\alpha^{n-1} + \cdots + b_{n-k}\alpha^{n-k}$$

$$\alpha^k - b_{n-1}\alpha^{k-1} - \cdots - b_{n-k+1}\alpha - b_{n-k} = 0.$$

Let $\alpha_1, \ldots, \alpha_k$ be the roots.

Might not be distinct.

If α_i appears L times then
α_i^n, $n\alpha_i^n$, $n^2\alpha_i^n$, …, $n^{L-1}\alpha_i^n$ are solutions.
Why We will not to into this but it involves that any multiple root of $p(x)$ is also a root of $p'(x)$.

Its all about the Base, bout the Base...

We now have n **different** solutions which we will call:
$n^{j_1}\alpha_1^n$, $n^{j_2}\alpha_2^n$, ..., $n^{j_k}\alpha_k^n$,
(most of the j's are 0).
For $0 \le L \le k-1$:
Use a_L:

$$c_1 L^{j_1}\alpha_1^L + \cdots + c_k L^{j_k}\alpha_k^L = a_L$$

Its all about the Base, bout the Base...

We now have n **different** solutions which we will call:
$n^{j_1}\alpha_1^n$, $n^{j_2}\alpha_2^n$, ..., $n^{j_k}\alpha_k^n$,
(most of the j's are 0).
For $0 \le L \le k-1$:
Use a_L:

$$c_1 L^{j_1}\alpha_1^L + \cdots + c_k L^{j_k}\alpha_k^L = a_L$$

This gives k linear equations in k variables.

Its all about the Base, bout the Base...

We now have n **different** solutions which we will call:
$n^{j_1}\alpha_1^n$, $n^{j_2}\alpha_2^n$, ..., $n^{j_k}\alpha_k^n$,
(most of the j's are 0).
For $0 \le L \le k-1$:
Use a_L:

$$c_1 L^{j_1}\alpha_1^L + \cdots + c_k L^{j_k}\alpha_k^L = a_L$$

This gives k linear equations in k variables.

That can be solved.

Its all about the Base, bout the Base...

We now have n **different** solutions which we will call:
$n^{j_1}\alpha_1^n$, $n^{j_2}\alpha_2^n$, ..., $n^{j_k}\alpha_k^n$,
(most of the j's are 0).
For $0 \le L \le k-1$:
Use a_L:

$$c_1 L^{j_1}\alpha_1^L + \cdots + c_k L^{j_k}\alpha_k^L = a_L$$

This gives k linear equations in k variables.

That can be solved.

Then you have the closed form solution.

More Complicated Recurrences

An Example

Recall the Emily Sequence:
$a_0 = 0$
$a_1 = 1$
$(\forall n \geq 2)[a_n = 5a_{n-1} - 6a_{n-2} + n^2]$.

An Example

Recall the Emily Sequence:

$a_0 = 0$

$a_1 = 1$

$(\forall n \geq 2)[a_n = 5a_{n-1} - 6a_{n-2} + n^2]$.

How to solve this?

An Example

Recall the Emily Sequence:
$a_0 = 0$
$a_1 = 1$
$(\forall n \geq 2)[a_n = 5a_{n-1} - 6a_{n-2} + n^2]$.

How to solve this?

Lemma Assume
$f(n)$ is a solution to $a_n = 5a_{n-1} - 6a_{n-2} + n^2$, and

An Example

Recall the Emily Sequence:

$a_0 = 0$

$a_1 = 1$

$(\forall n \geq 2)[a_n = 5a_{n-1} - 6a_{n-2} + n^2]$.

How to solve this?

Lemma Assume

$f(n)$ is a solution to $a_n = 5a_{n-1} - 6a_{n-2} + n^2$, and

$g(n)$ is a solution to $a_n = 5a_{n-1} - 6a_{n-2}$.

An Example

Recall the Emily Sequence:
$a_0 = 0$
$a_1 = 1$
$(\forall n \geq 2)[a_n = 5a_{n-1} - 6a_{n-2} + n^2]$.

How to solve this?

Lemma Assume
$f(n)$ is a solution to $a_n = 5a_{n-1} - 6a_{n-2} + n^2$, and
$g(n)$ is a solution to $a_n = 5a_{n-1} - 6a_{n-2}$.
Then $f(n) + g(n)$ is a solution to
$a_n = 5a_{n-1} - 6a_{n-2} + n^2$.

An Example

Recall the Emily Sequence:

$a_0 = 0$

$a_1 = 1$

$(\forall n \geq 2)[a_n = 5a_{n-1} - 6a_{n-2} + n^2]$.

How to solve this?

Lemma Assume

$f(n)$ is a solution to $a_n = 5a_{n-1} - 6a_{n-2} + n^2$, and

$g(n)$ is a solution to $a_n = 5a_{n-1} - 6a_{n-2}$.

Then $f(n) + g(n)$ is a solution to

$a_n = 5a_{n-1} - 6a_{n-2} + n^2$.

The proof is algebra which we will skip.

Ignore The Base Case for a While

Plan For now find **all** solutions to

$$a_n = 5a_{n-1} - 6a_{n-2} \text{ called } \textit{The Homogenous Solution}$$

Ignore The Base Case for a While

Plan For now find **all** solutions to

$$a_n = 5a_{n-1} - 6a_{n-2} \text{ called } \textit{The Homogenous Solution}$$

and **some** solution to

$$a_n = 5a_{n-1} - 6a_{n-2} + n^2 \text{called } \textit{The Particular Solution}$$

Ignore The Base Case for a While

Plan For now find **all** solutions to

$$a_n = 5a_{n-1} - 6a_{n-2} \text{ called } \textit{The Homogenous Solution}$$

and **some** solution to

$$a_n = 5a_{n-1} - 6a_{n-2} + n^2 \text{called } \textit{The Particular Solution}$$

We will then add them.

Finding the Homogenous Solution

We want all solutions to $a_n = 5a_{n-1} - 6a_{n-2}$.

Finding the Homogenous Solution

We want all solutions to $a_n = 5a_{n-1} - 6a_{n-2}$.

This we know how to do so I will just give you the answer:

$$c2^n + d3^n \text{ where } c, d \in R \ .$$

Finding the Homogenous Solution

We want all solutions to $a_n = 5a_{n-1} - 6a_{n-2}$.

This we know how to do so I will just give you the answer:

$$c2^n + d3^n \text{ where } c, d \in R \ .$$

We will save finding c, d for the base case.

Finding a Particular Solution

We want a solutions to $a_n = 5a_{n-1} - 6a_{n-2} + n^2$.

Finding a Particular Solution

We want a solutions to $a_n = 5a_{n-1} - 6a_{n-2} + n^2$.

Guess it is of the form $An^2 + Bn + C$.

Finding a Particular Solution

We want a solutions to $a_n = 5a_{n-1} - 6a_{n-2} + n^2$.

Guess it is of the form $An^2 + Bn + C$.

$$An^2+Bn+C = 5(A(n-1)^2+B(n-1)+C)-6(A(n-2)^2+B(n-2)+C)+n^2$$

Finding a Particular Solution

We want a solutions to $a_n = 5a_{n-1} - 6a_{n-2} + n^2$.
Guess it is of the form $An^2 + Bn + C$.

$$An^2+Bn+C = 5(A(n-1)^2+B(n-1)+C)-6(A(n-2)^2+B(n-2)+C)+n^2$$

$$An^2 + Bn + C = (1-A)n^2 + (14A+2B)n + (5A-3B+2C-24)$$

Finding a Particular Solution

We want a solutions to $a_n = 5a_{n-1} - 6a_{n-2} + n^2$.
Guess it is of the form $An^2 + Bn + C$.

$$An^2+Bn+C = 5(A(n-1)^2+B(n-1)+C)-6(A(n-2)^2+B(n-2)+C)+n^2$$

$$An^2 + Bn + C = (1-A)n^2 + (14A+2B)n + (5A-3B+2C-24)$$

GOTO next slide

Finding a Particular Solution (cont)

For ALL n we have:

$$An^2 + Bn + C = (1 - A)n^2 + (14A + 2B)n + (5A - 3B + 2C - 24)$$

We match coefficients.

Finding a Particular Solution (cont)

For ALL n we have:

$$An^2 + Bn + C = (1 - A)n^2 + (14A + 2B)n + (5A - 3B + 2C - 24)$$

We match coefficients.

$A = 1 - A$, so $A = \frac{1}{2}$.

Finding a Particular Solution (cont)

For ALL n we have:

$$An^2 + Bn + C = (1 - A)n^2 + (14A + 2B)n + (5A - 3B + 2C - 24)$$

We match coefficients.

$A = 1 - A$, so $A = \frac{1}{2}$.

$B = 14A + 2B$, so $B = -14A = -7$.

Finding a Particular Solution (cont)

For ALL n we have:

$$An^2 + Bn + C = (1 - A)n^2 + (14A + 2B)n + (5A - 3B + 2C - 24)$$

We match coefficients.

$A = 1 - A$, so $A = \frac{1}{2}$.

$B = 14A + 2B$, so $B = -14A = -7$.

$C = 5A - 3B + 2C - 24 = \frac{5}{2} + 21 + 2C - 24 = 2C - 3$

Finding a Particular Solution (cont)

For ALL n we have:

$$An^2 + Bn + C = (1 - A)n^2 + (14A + 2B)n + (5A - 3B + 2C - 24)$$

We match coefficients.

$A = 1 - A$, so $A = \frac{1}{2}$.

$B = 14A + 2B$, so $B = -14A = -7$.

$C = 5A - 3B + 2C - 24 = \frac{5}{2} + 21 + 2C - 24 = 2C - 3$

So $C = 3$.

Finding a Particular Solution (cont)

For ALL n we have:

$$An^2 + Bn + C = (1 - A)n^2 + (14A + 2B)n + (5A - 3B + 2C - 24)$$

We match coefficients.

$A = 1 - A$, so $A = \frac{1}{2}$.

$B = 14A + 2B$, so $B = -14A = -7$.

$C = 5A - 3B + 2C - 24 = \frac{5}{2} + 21 + 2C - 24 = 2C - 3$

So $C = 3$.

Particular solution is

$$-\frac{1}{2}n^2 - 7n + 3$$

Its all about the Base, bout the Base...

For any constants c, d

$$c \times 2^n + d \times 3^n - \frac{1}{2}n^2 - 7n + 3$$

satisfy the recurrence.
We use the base case to find c, d that work.

Its all about the Base, bout the Base...

For any constants c, d

$$c \times 2^n + d \times 3^n - \frac{1}{2}n^2 - 7n + 3$$

satisfy the recurrence.

We use the base case to find c, d that work.

We leave this to the reader.

How to Guess the Particular Solution

If the extra term is a poly of degree d, guess a poly of degree d

How to Guess the Particular Solution

If the extra term is a poly of degree d, guess a poly of degree d

If the extra term is 5^n then guess $A \times 5^n$.

How to Guess the Particular Solution

If the extra term is a poly of degree d, guess a poly of degree d

If the extra term is 5^n then guess $A \times 5^n$.

If the extra term is BLAH then guess something of BLAH form with undetermined coefficients.

General Algorithm

We are not going to present the general algorithm for a general recurrence with an extra term since we are confident you an do that yourself.

Deja Vu Now or Later

If you have had a course in Differential Equations then you may be feeling a sense of Deja Vu.

Deja Vu Now or Later

If you have had a course in Differential Equations then you may be feeling a sense of Deja Vu.

The technique described is very similar to techniques to solve differential equations of the form

$$y'' + ay' + by + f(x) = 0$$

Deja Vu Now or Later

If you have had a course in Differential Equations then you may be feeling a sense of Deja Vu.

The technique described is very similar to techniques to solve differential equations of the form

$$y'' + ay' + by + f(x) = 0$$

If you have a course in Diff Equations later then when you take it you will have a sense of Deja Vu, because of this course.

BILL AND EMILY RECORD LECTURE!!!!

UNTIMED PART OF FINAL IS MONDAY May 10 9:00AM. NO DEAD CAT

FINAL IS MONDAY May 17 8:00PM-10:15PM

FILL OUT COURSE EVALS for ALL YOUR COURSES!!!