The Roots Hierarchy

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1 Introduction

The main proof in this note is from *Problems from the Book* by Dospinescu and Andreescu.

We want to classify real numbers in terms of their complexity.

Def 1.1 Let $d \in \mathsf{N}$.

1. $\mathsf{Z}_d[x]$ is the set of polynomials of degree d over Z (the integers).

2. roots_d is the set of roots of polynomials in $\mathsf{Z}_d[x]$. Note that $\text{roots}_1 = \mathsf{Q}$.

Clearly $\text{roots}_1 \subseteq \text{roots}_2 \subseteq \text{roots}_3 \subseteq \cdots$

We want to show that $\text{roots}_1 \subset \text{roots}_2 \subset \text{roots}_3 \subset \cdots$

2 The Hierarchy is Proper

We show that $\text{roots}_2 \subset \text{roots}_3$. All of the ideas to show $\text{roots}_{d-1} \subset \text{roots}_d$ are contained in the proof. The main method for the proof is taken from chapter 9 of *Problems from the Book* by Titu Andreescu and Gabriel Dospinescu.

Theorem 2.1 $\text{roots}_2 \subset \text{roots}_3$.

Proof: Clearly $\text{roots}_2 \subseteq \text{roots}_3$. We show that $2^{1/3} \in \text{roots}_3 - \text{roots}_2$ which implies

$$\text{roots}_2 \subset \text{roots}_3.$$

Clearly $2^{1/3}$ is a root of $x^3 - 2 = 0$ and hence $2^{1/3} \in \text{roots}_3$. We show that $2^{1/3} \notin \text{roots}_2$

Assume, by way of contradiction, that there exists $a_0, a_1, a_2 \in \mathsf{Z}$ such that

$$a_2(2^{1/3})^2 + a_1(2^{1/3}) + a_0 = 0$$

which is

$$a_2 \times 2^{2/3} + a_1 \times 2^{1/3} + a_0 \times 1 = 0$$

We assume the following about (a_2, a_1, a_0): *They are not all even.* If they are then divide each one by 2 to get a smaller poly over Z and use that.

Multiply this equation by 1, $2^{1/3}, 2^{2/3}$ to get

$$a_2 \times 2^{2/3} + a_1 \times 2^{1/3} + a_0 \times 1 = 0$$

$$2a_2 \times 1 + a_1 \times 2^{2/3} + a_0 \times 2^{1/3} = 0$$

$$2a_2 \times 2^{1/3} + 2a_1 \times 1 + a_0 \times 2^{2/3} = 0$$

We rewrite these putting the powers of 2 in order.

$$a_2 \times 2^{2/3} + a_1 \times 2^{1/3} + a_0 \times 1 = 0$$

$$a_1 \times 2^{2/3} + a_0 \times 2^{1/3} + 2a_2 \times 1 = 0$$

$$a_0 \times 2^{2/3} + 2a_2 \times 2^{1/3} + 2a_1 \times 1 = 0$$

We rewrite this as a matrix times a vector being the zero vector:

$$\begin{pmatrix} a_2 & a_1 & a_0 \\ a_1 & a_0 & 2a_2 \\ a_0 & 2a_2 & 2a_1 \end{pmatrix} \begin{pmatrix} 2^{2/3} \\ 2^{1/3} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Let

$$A = \begin{pmatrix} a_2 & a_1 & a_0 \\ a_1 & a_0 & 2a_2 \\ a_0 & 2a_2 & 2a_1 \end{pmatrix}$$

The matrix above can be multiplied by a non-zero vector and get zero. Hence the matrix has det 0. Hence the det is 0 MOD 2. Hence the det of

$$A = \begin{pmatrix} a_2 & a_1 & a_0 \\ a_1 & a_0 & 0 \\ a_0 & 0 & 0 \end{pmatrix}$$

mod 2 is 0. By the expansion of det, $a_0^3 \equiv 0 \pmod 2$, so $a_0 \equiv 0 \pmod 2$.

Hence the det of

$$A = \begin{pmatrix} a_2 & a_1 & 0 \\ a_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

mod 2 is 0.

We rewrite the matrix as

$$A = \begin{pmatrix} a_2 & a_1 & 2b_0 \\ a_1 & 2b_0 & 2a_2 \\ 2b_0 & 2a_2 & 2a_1 \end{pmatrix}$$

Since this matrix has det 0, so does the matrix obtained by dividing the last column by 2:

$$A = \begin{pmatrix} a_2 & a_1 & b_0 \\ a_1 & 2b_0 & a_2 \\ 2b_0 & 2a_2 & a_1 \end{pmatrix}$$

Now take this matrix mod 2 to get

$$A = \begin{pmatrix} a_2 & a_1 & b_0 \\ a_1 & 0 & a_2 \\ 0 & 0 & a_1 \end{pmatrix}$$

We find the det mod 2 by expanding around the middle of the top row a_1:

$$a_1((a_1^2 + a_2 \times 0) + 0 + 0) = a_1^3.$$

Hence $a_1^3 \equiv 0 \pmod{2}$, so $a_1 \equiv 0 \pmod{2}$.

We rewrite our matrix as

$$A = \begin{pmatrix} a_2 & 2b_1 & b_0 \\ 2b_1 & 2b_0 & a_2 \\ 2b_0 & 2a_2 & 2b_1 \end{pmatrix}$$

We divide the middle column by 2 to get

$$A = \begin{pmatrix} a_2 & b_1 & b_0 \\ 2b_1 & b_0 & a_2 \\ 2b_0 & a_2 & 2b_1 \end{pmatrix}$$

We take this matrix mod 2 to get

$$A = \begin{pmatrix} a_2 & b_1 & b_0 \\ 0 & b_0 & a_2 \\ 0 & a_2 & 0 \end{pmatrix}$$

By expanding on the upper left a_2 we get $a_2^3 \equiv 0$ so $a_2 \equiv 0$.

Hence a_0, a_1, a_2 are all even, which is a contradiction.

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