START RECORDING

Constructive Induction

CMSC 250

Introductory Example

We already know that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$$

- But how? Who told us this?
- This is **not** how math works; we usually do **not** know the answer ahead of time!

Making a Good Guess with Calculus

- Calculus tells us that (discrete) sums are approximations of (continuous) integrals.
- Then, we can observe that:

$$\sum_{i=1}^{n} i \approx \int_{1}^{n} x \, dx = \frac{1}{2}n^2 + c, \qquad c \in \mathbb{R}$$

• So we know that the sum ought to be *some* quadratic function of n.

Making a Good Guess with CS

- Another way to guess the quadratic form would be with plotting!
- Suppose $f(n) = \sum_{i=1}^{n} i$. Then:

•
$$f(0) = \sum_{i=1}^{0} i = 0$$

•
$$f(1) = \sum_{i=1}^{1} i = 1$$

•
$$f(2) = \sum_{i=1}^{2} i = 1 + 2 = 3$$

•
$$f(3) = \sum_{i=1}^{3} i = 1 + 2 + 3 = 6$$

• ...

•
$$f(30) = \sum_{i=1}^{30} i = 1 + 2 + \dots + 30 = 465$$

• We can then fit a curve and see the quadratic curve by ourselves!

Making a Good Guess

- We saw that the sum is some quadratic polynomial. This is all we know!
- So $\sum_{i=1}^{n} i$ is some poly(n) with degree 2, i.e

$$\sum_{i=1}^{n} i = An^2 + Bn + C, \qquad A, B, C \in \mathbb{R}$$

How to determine A, B, and C?

General Logic

- Solve as if you had an inductive proof (so IB, IH, IS)
- For every step, we will establish **conditions** on A, B,C **such that** the relevant step is correct.
 - Contrast this with directly proving that every step is correct.

Constant C

• IB: LHS is $\sum_{i=1}^{0} i = 0$. For RHS to be equal to LHS we need:

$$An^2 + Bn + C = 0 \Rightarrow C = 0$$

• So we already know that C=0.

• IH: Assume that the proposition holds for $n \geq 0$. Then:

$$\sum_{i=1}^{n} i = An^2 + Bn$$

IS: We want to prove that

$$\left(\sum_{i=1}^{n} i = An^2 + Bn\right) \Rightarrow \left(\sum_{i=1}^{n+1} i = A(n+1)^2 + B(n+1)\right)$$

• IH: Assume that the proposition holds for $n \geq 0$. Then:

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$$P(n)$$

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$$P(n)$$

$$P(n+1)$$

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1) \stackrel{\mathsf{IH}}{=} An^2 + Bn + (n+1)$$

• We have to equate this to $A(n + 1)^2 + B(n + 1)$, since this is what we're trying to prove:

$$An^{2} + Bn + (n + 1) = A(n + 1)^{2} + B(n + 1) \Rightarrow$$

 $An^{2} + Bn + (n + 1) = An^{2} + 2An + A + Bn + B \Rightarrow$
 $n + 1 = 2An + (A + B)$

$$n+1=2An+(A+B)$$

• This is an equality between polynomials of k, so equating the coefficients yields:

$$1 = 2A$$
$$A + B = 1$$

$$n+1=2An+(A+B)$$

• This is an equality between polynomials in n, so equating the coefficients yields:

$$1 = 2A$$
$$A + B = 1$$

 Note: The IS did not end up with TRUE, but with conditions on A,B for it to be TRUE.

All Our Constraints

1.
$$C = 0$$

2.
$$A+B=1$$

3.
$$2 \cdot A = 1$$

• Algebra yields $A = B = \frac{1}{2}$, so:

$$\sum_{i=0}^{n} i = \frac{1}{2}n^2 + \frac{1}{2}n + 0 = \frac{n(n+1)}{2}$$

What if Our Guess is Wrong (Over)?

1. Suppose we guess

$$\sum_{i=1}^{n} i = A \cdot n^3 + B \cdot n^2 + C \cdot n + D$$

2. This still works, we will just find A = 0 (try it at home!)

What if Our Guess is Wrong (Under)?

1. Suppose we guess

$$\sum_{i=1}^{n} i = A \cdot n + B$$

2. This does not work (infeasible equation), no $A, B \in \mathbb{R}$ will satisfy the constraints (try it at home!)

Another Example (with Bounds!)

• Let a be a sequence defined as follows:

$$a_n = \begin{cases} 2, & n = 0 \\ 50, & n = 1 \\ 10a_{n-1} + 3a_{n-2}, & n \ge 2 \end{cases}$$

• Task: Find an upper bound for a_n .

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- What kind of inductive structure am I expecting?

Weak

Strong

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- What kind of inductive structure am I expecting?



An inductive base with > 1 elements and a recursive rule with references to two prior terms hints towards strong induction...

Key Step

$$a_n = \begin{cases} 2, & n = 0 \\ 50, & n = 1 \\ 10a_{n-1} + 3a_{n-2}, n \ge 2 \end{cases}$$

• Because of our experience with sequences like Fibonacci, Tribonacci that all have this form, we **suspect**:

$$a_n \leq C \cdot D^n$$
, $C, D \in \mathbb{R}$

Constraints on C

- IB:
 - $a_0 \le C \cdot D^0 \Leftrightarrow 2 \le C$
 - $a_1 \le C \cdot D^1 \Leftrightarrow 50 \le C \cdot D$

Inductive Hypothesis

- IB:
 - $a_0 \le C \cdot D^0 \Leftrightarrow 2 \le C$
 - $a_1 \le C \cdot D^1 \Leftrightarrow 50 \le C \cdot D$
- IH: Let $n \ge 1$. Assume that $(\forall i \in \{0, 1, 2, ... n\})[a_i \le C \cdot D^i]$

Inductive Step

- IB:
 - $a_0 \le C \cdot D^0 \Leftrightarrow 2 \le C$
 - $a_1 \le C \cdot D^1 \Leftrightarrow 50 \le C \cdot D$
- IH: Let $n \ge 1$. Assume that $\forall i \in \{0, 1, 2, ..., n\}, a_i \le C \cdot D^i$.
- IS:

$$(\forall i \in \{0, 1, 2, \dots n\})[a_i \le C \cdot D^i] \Rightarrow \left(a_{n+1} \le C \cdot D^{n+1}\right)$$

Inductive Step

• IS:

$$(\forall i \in \{0, 1, 2, \dots n\})[a_i \le C \cdot D^i] \Rightarrow (a_{n+1} \le C \cdot D^{n+1})$$

• From the definition of a, we have $a_{n+1} = 10a_n + 3a_{n-1}$. Therefore,

$$a_{n+1} = 10a_n + 3a_{n-1} \le 10 \cdot C \cdot D^n + 3 \cdot C \cdot D^{n-1}$$
 (By IH)

• Want $10 \cdot C \cdot D^n + 3 \cdot C \cdot D^{n-1} \leq C \cdot D^{n+1}$

Inductive Step

Want

$$10 \cdot \cancel{\mathcal{L}} \cdot D^n + 3 \cdot \cancel{\mathcal{L}} \cdot D^{n-1} \le \cancel{\mathcal{L}} \cdot D^{n+1} \Leftrightarrow 10 \cdot D^n + 3 \cdot D^{n-1} \le D^{n+1}$$

• Dividing both sides by D^{n-1} yields:

$$10D + 3 \le D^2$$

- 1. $2 \le C$
- $2.50 \leq C \cdot D$
- 3. $10D + 3 \le D^2$
- We deal with constraint 3 first.
 - Smallest $D \in \mathbb{R}^{>0}$ that satisfies it:

- 1. $2 \le C$
- 2. $50 \le C \cdot D$
- 3. $10D + 3 \le D^2$
- We deal with constraint 3 first.
 - Smallest $D \in \mathbb{R}^{>0}$ that satisfies it: NO, WE ARE BUSY PEOPLE AND WE DON'T WANT TO SPEND TIME SOLVING $D^2-10D-3 \geq 0$
 - Smallest $D \in \mathbb{N}$ that satisfies it: $D = \cdots$??? (FIND ONE REAL QUICK, PLZ)

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D = 11 works!

- 1. $2 \le C$
- $2.50 \leq C \cdot D$
- 3. $10D + 3 \le D^2$
- Constraint (3) satisfied when $D \ge 11$ (just discussed)
- Since we want to find **tight** bounds for a_n , to minimize C, we select
- D=11 and from constraint (2) we have: $50 \le C \cdot 11 \Leftrightarrow C \ge 4.55 \Rightarrow C_{min}=4.55$

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- D=11 and from constraint (2) we have: $50 \le C \cdot 11 \Leftrightarrow C \ge 4.55 \Rightarrow C_{min}=4.55$
- Conclusion:

$$a_n \le 4.55 \cdot 11^n$$

Work on This

A slight modification on the previous sequence:

$$a_n = \begin{cases} 10, & n = 0 \\ 50, & n = 1 \\ 10a_{n-1} + 3a_{n-2}, & n \ge 2 \end{cases}$$

• Assuming that we still suspect $a_n \le C \cdot D^n$, you solve for the new C, D right now!

Work on This

A slight modification on the previous sequence:

$$a_n = \begin{cases} 10, & n = 0 \\ 50, & n = 1 \\ 10a_{n-1} + 3a_{n-2}, n \ge 2 \end{cases}$$

- Assuming that we still suspect $a_n \leq C \cdot D^n$, solve for the new C, D!
- Your solution ought to be C = 10, D = 11. What do you observe?

Coin Problem

- In Celestia, there are only 7c and 10c coins.
- We want to find the *least monetary amount* payable exclusively with such coins!
- In quantifiers (all quantifications assumed over N)

$$(\forall n \ge A)(\exists n_1, n_2)[n = 7n_1 + 10n_2]$$

- Goal: Find constraints on A via constructive induction!
- IB: ???

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- Goal: Find constraints on A via constructive induction!
- IB: Defer for later!!!
- IH: Assume that for $n \ge A$, $(\exists n_1, n_2)[n = 7 \cdot n_1 + 10n_2]$

Coin Problem (IS)

- From the IH we have $(\exists n_1, n_2)[n = 7 \cdot n_1 + 10n_2]$
- How can we add/remove coins to get another cent?

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- How can we add/remove coins to get another cent?
 - 1. $n_2 \ge 2$: Remove two 10c coins, add three 7c coins

$$n + 1 = 7n_1 + 10n_2 + 1 = 7n_1 + 10n_2 + (21 - 20)$$

= $7(n_1 + 3) + 10(n_2 - 2)$

- From the IH we have $(\exists n_1, n_2)[n = 7 \cdot n_1 + 10n_2]$
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= $7(n_1 + 3) + 10(n_2 - 2)$

2. $n_1 \ge 7$: Remove seven 7c coins, add five 10c coins

$$n + 1 = 7n_1 + 10n_2 + 1 = 7n_1 + 10n_2 + (50 - 49)$$

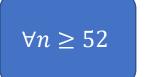
= $7(n_1 - 7) + 10(n_2 + 5)$

3. $(n_1 \le 6) \land (n_2 \le 1)$: Max value is $6 \times 7 + 1 \times 10 = 52$, so $n \le 52$.

• We've shown that if $n \ge 53$, then

$$((\exists n_1, n_2)[n = 7 \cdot n_1 + 10n_2]) \Rightarrow ((\exists n_1, n_2)[n + 1 = 7 \cdot n_1 + 10n_2])$$

• For which n do we know that $((\exists a, b \in \mathbb{N})[n = 7a + 10b]$?



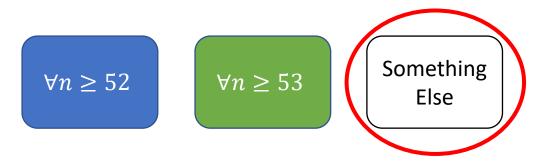
 $\forall n \geq 53$

Something Else

• We've shown that if $n \ge 53$, then

$$((\exists n_1, n_2)[n = 7 \cdot n_1 + 10n_2]) \Rightarrow ((\exists n_1, n_2)[n + 1 = 7 \cdot n_1 + 10n_2])$$

• For which n do we know that $((\exists a, b \in \mathbb{N})[n = 7a + 10b]$?



Only the implication holds! We don't have any **hard truth** (base) about whether it EVER holds.

3. $(n_1 \le 6) \land (n_2 \le 1)$: Max value is $6 \times 7 + 1 \times 10 = 52$, so $n \le 52$.

- Condition: $A \geq 53$.
- Now I need a base case.
- $(\exists ? n_1, n_2 \in \mathbb{N})[53 = 7 \cdot n_1 + 10n_2]$



3. $(n_1 \le 6) \land (n_2 \le 1)$: Max value is $6 \times 7 + 1 \times 10 = 52$, so $k \le 52$.

- Condition: $A \geq 53$.
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Prove it at home (use cases)

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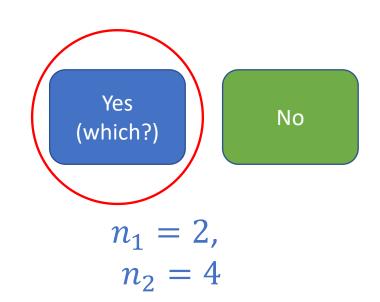
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No

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• We've shown that if $n \ge 53$, then

$$((\exists n_1, n_2)[n = 7 \cdot n_1 + 10n_2]) \Rightarrow ((\exists n_1, n_2)[n + 1 = 7 \cdot n_1 + 10n_2])$$

• We've also shown that $(\exists r_1, r_2 \in \mathbb{N})[54 = 7r_1 + 10r_2]$ $(r_1 = 2, r_2 = 4)$

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$$((\exists n_1, n_2)[n = 7 \cdot n_1 + 10n_2]) \Rightarrow ((\exists n_1, n_2)[n + 1 = 7 \cdot n_1 + 10n_2])$$

- We've also shown that $(\exists r_1, r_2 \in \mathbb{N})[54 = 7r_1 + 10r_2]$ $(r_1 = 2, r_2 = 4)$
- What do we know NOW about the theorem?

True for $n \ge 52$

True for $n \ge 53$

True for $n \ge 54$

Nothing

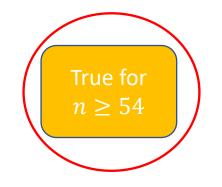
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- We've also shown that $(\exists r_1, r_2 \in \mathbb{N})[54 = 7r_1 + 10r_2]$ $(r_1 = 2, r_2 = 4)$
- What do we know NOW about the theorem?

True for $n \ge 52$

True for $n \ge 53$



Nothing

What is A?

• Recall the theorem (all quantifiers over $\mathbb N$):

$$(\forall n \ge A)(\exists n_1, n_2)[n = 7n_1 + 10n_2]$$

- Our goal was to find A.
- A = 54 works, and is optimal, since A = 53 does not work.

Question

• Is the theorem true for any $n \le 53$?

Yes (which?)

No (Why?)

Question

• Is the theorem true for any $n \leq 53$?



0, 7, 10, 14, 17, 20, 21, 24, 27, 28, 30, 31, 34, 35, 37, 38, 40, 41, 42, 44, 45, 47, 48, 49, 50, 51, 52

Note that there are gaps between these integers!

• Let *a* be a sequence defined as follows:

$$a_n = \begin{cases} 0, & n = 0 \\ 2, & n = 1 \\ a_{\left\lfloor \frac{n}{2} \right\rfloor} + a_{\left\lfloor \frac{n}{4} \right\rfloor} + 5n, & n \ge 2 \end{cases}$$

• Then, find $C \in \mathbb{R}$ such that

$$(\forall n \in \mathbb{N})[a_n \leq C \cdot n]$$

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Recursions like this have linear upper bounds

• Let *a* be a sequence defined as follows:

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• Then, find $C \in \mathbb{R}$ such that

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Recursions like this have linear upper bounds

• We proceed via **strong induction** on n.

• Let *a* be a sequence defined as follows:

$$a_n = \begin{cases} 0, & n = 0 \\ 2, & n = 1 \\ a_{\left\lfloor \frac{n}{2} \right\rfloor} + a_{\left\lfloor \frac{n}{4} \right\rfloor} + 5n, & n \ge 2 \end{cases}$$

• Then, find $C \in \mathbb{R}$ such that

$$(\forall n \in \mathbb{N})[a_n \le C \cdot n]$$

- We proceed via **strong induction** on n.
- In fact, to make some of the math easier, we will assume the hypothesis until P(n-1) and prove the step for P(n) instead of P(n+1)

- IB:
 - For n=0, $a_0 \le C \cdot 0 \Leftrightarrow 0 \le 0$. No constraints on C yet!
 - For n=1, $a_1 \le C \cdot n \Leftrightarrow 2 \le C$. Done. We have our first lower bound for C.

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- IH: Let $n \ge 2$. Then, assume $(\forall i \in \{0,1,2,...,n-1\}[P(i)], \text{ where } P(i) \text{ means } a_i \le C \cdot i$

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 - For n=0, $a_0 \le C \cdot 0 \Leftrightarrow 0 \le 0$. No constraints on C yet!
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- IH: Let $n \ge 2$. Then, assume $(\forall i \in \{0,1,2,...,n-1\}[P(i)], \text{ where } P(i) \text{ means } a_i \le C \cdot i$
- IS: We attempt to prove $(P(0) \land P(1) \land P(2) \land \cdots \land P(n-1)) \Rightarrow P(n)$:

$$\bigwedge_{i=0}^{i=n-1} (a_i \le C \cdot i) \Rightarrow a_n \le C \cdot n$$

• IS: We attempt to prove $(P(1) \land P(2) \land \cdots \land P(n-1)) \Rightarrow P(n)$:

$$\bigwedge_{i=0}^{i=n-1} (a_i \le C \cdot i) \Rightarrow a_n \le C \cdot n$$

• From the IH, and taking into consideration that $0 \le \left\lfloor \frac{n}{4} \right\rfloor$, $\left\lfloor \frac{n}{2} \right\rfloor \le n$, we have (next slide):

• From the IH, and taking into consideration that $0 \le \left\lfloor \frac{n}{4} \right\rfloor$, $\left\lfloor \frac{n}{2} \right\rfloor \le n$, we have:

$$\begin{cases} a_{\lfloor n/4 \rfloor} \le C \cdot \lfloor n/4 \rfloor \le C \cdot \frac{n}{4} \\ a_{\lfloor n/2 \rfloor} \le C \cdot \lfloor n/2 \rfloor \le C \cdot \frac{n}{2} \end{cases}$$

•
$$a_n = a_{\lfloor n/2 \rfloor} + a_{\lfloor n/4 \rfloor} + 5n \le C \cdot \frac{n}{2} + C \cdot \frac{n}{4} + 5n = \frac{n*(3C+20)}{4}$$

• We have:

$$a_n \le \frac{n*(3C+20)}{4}$$

• We want:

$$a_n \leq C \cdot n$$

• Hence, we want a C such that:

$$\frac{n*(3C+20)}{4} \le C \cdot n$$

$$\frac{\cancel{N}(3C + 20)}{4} \le C \cdot \cancel{N} \stackrel{n \ge 1}{\Leftrightarrow}$$

$$\frac{(3C + 20)}{4} \le C \Leftrightarrow$$

$$3C + 20 \le 4C \Leftrightarrow$$

$$C \ge 20$$

$$\Rightarrow C_{min} = 20$$

Constraints

- From the IB: $C \ge 2$
- From the IS: $C \ge 20$
- Since we want to minimize C, we set C = 20.

STOP RECORDING