An Interesting Sum

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$$\frac{1}{1\times 2}+\frac{1}{1\times 3}+\frac{1}{1\times 4}+\frac{1}{2\times 3}+\frac{1}{2\times 4}+\frac{1}{3\times 4}+$$

$$\frac{1}{1\times 2\times 3}+\frac{1}{1\times 2\times 4}+\frac{1}{1\times 3\times 4}+\frac{1}{2\times 3\times 4}+\frac{1}{1\times 2\times \times 3\times 4}=4.$$

Notation

Let $n \in \mathbb{N}$. Let S_n be the **multiset**

$$S_n = \bigcup_{k=1}^{n} \{a_1 \cdots a_k \colon a_1, \ldots, a_k \in \{1, \ldots, n\} \wedge (\forall i, j)[a_i \neq a_j]\}.$$

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$$S_1 = \{1\}$$

$$S_2 = \{1, 2, 1 \times 2\} = \{1, 2, 2\}.$$

$$S_3 = \{1, 2, 3, 1 \times 2, 1 \times 3, 2 \times 3, 1 \times 2 \times 3\} = \{1, 2, 3, 2, 3, 6, 6\}.$$

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$$\sum_{x\in S_4}\frac{1}{x}=4.$$

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To prove this we first need a lemma.

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(a) $S_n = S_{n-1} \cup n \cdot S_{n-1} \cup \{n\}$.
(b) $\sum_{x \in S_n} \frac{1}{x} = \sum_{x \in S_{n-1}} \frac{1}{x} + \sum_{x \in S_{n-1}} \frac{1}{nx} + \frac{1}{n}$.

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Putting this all together you get the lemma.

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$$n - 1 + \frac{1}{n}(n-1) + \frac{1}{n} = n - 1 + n - \frac{1}{n} + \frac{1}{n} = n.$$