Duplicator Spoiler Games

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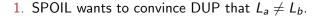
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We will call SPOIL S and DUP D to fit on slides.

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Bill plays a student $(L_3, L_4, 2)$, $(L_3, L_4, 3)$

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- 1. S beats D in the (L_a, L_b, k) game.
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- 3. GENERALLY: Who wins (L_a, L_b, k) .

Can use any orderings L, L^{\prime}

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Play a student $\mathbb N$ and $\mathbb Z$ with 1 move, 2 moves

In all problems we want a k such that condition holds.

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- **4**. D wins $(L_{10}, \mathbb{N} + \mathbb{N}^*, k 1)$, S wins $(L_{10}, \mathbb{N} + \mathbb{N}^*, k)$.

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- 5. D wins $(\mathbb{N} + \mathbb{Z}, \mathbb{N}, k 1)$, S wins $(\mathbb{N} + \mathbb{Z}, \mathbb{N}, k)$.

A Notion of L, L' being Similar

Let L and L' be two linear orderings.

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Let L and L' be two linear orderings. **Def** If D wins the k-round DS-game on L, L' then L, L' are k-game equivalent (denoted $L \equiv_k^G L'$).

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- 1. $\mathbb{Q} \models \phi$
- 2. $\mathbb{N} \models \neg \phi$

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If $Q \in \{\exists, \forall\}$ then
$$\operatorname{qd}((Qx_1)[\phi(x_1, \dots, x_n)] = \operatorname{qd}(\phi_1(x_1, \dots, x_n)) + 1.$$

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$$\operatorname{qd}((\forall x)(\forall z)[x < z \to (\exists y)[x < y < z]]) = 2 + 1 = 3$$

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Let L and L' be two linear orderings. **Def** L and L' are k-truth-equiv $(L \equiv_k^T L')$

$$(\forall \phi, qd(\phi) \leq k)[L \models \phi \text{ iff } L' \models \phi.]$$

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- 1. $L \equiv_k^T L'$
- 2. $L \equiv_k^G L'$

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- 3. Upshot: Questions about expressability become questions about games.
- 4. Complexity: As Computer Scientists we think of complexity in terms of time or space (e.g., sorting n elements can be done in roughly n log n comparisons). But how do you measure complexity for concepts where time and space do not apply? One measure is quantifier depth. These games help us prove LOWER BOUNDS on quantifier depth!

Proving DUP Wins Rigorously

Notation

The game where the orders are L and L', and its for n moves, will be denoted

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- Might not need induction on the smaller boards if they are orderings we already proved things about.

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$$(\mathbb{N} + \mathbb{N}^*, L_{2^n}; n-1)$$
 and $(\mathbb{N}, \mathbb{N}; n-1)$.

SP won't play on 2nd board. DUP wins 1st board by prior thm.

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SP won't play on 1st board. The 2nd board DUP wins by IH

2) If SP 1st move is x in \mathbb{Z} part of $\mathbb{N} + \mathbb{Z}$ then DUP plays 2^n in \mathbb{N} . The 2 games are $(\mathbb{N} + \mathbb{N}^*, L_{2^n}; n-1)$ and $(\mathbb{N}, \mathbb{N}; n-1)$.

SP won't play on 2nd board.

By last slide, on 2nd board DUP wins $(\mathbb{N}, \mathbb{N}; n-1)$.

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 and $\mathbb{Z} + \mathbb{Z}$

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SP 1st move is x is \mathbb{Z} . DUP picks x in first copy of \mathbb{Z} in $\mathbb{Z} + \mathbb{Z}$.

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IB n = 1. DUP clearly wins $(\mathbb{Z}, \mathbb{Z} + \mathbb{Z}; 1)$.

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You only have to do the cases that SP picks $x \in Z$.