START RECORDING

Mod Arithmetic

CMSC250

Divides

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- Examples:
 - 2|10
 - 5|25
 - 5 | 7
 - 0 \ 3
 - 8|8

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- THINK: Take large number *a*, divide by *m*, remainder is *b*
- Terminology: "Reducing a mod m"

\equiv VS \equiv

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- In Number Theory, $a \equiv b \pmod{m}$, read "a is *congruent to* $b \mod m$ ") means $m \mid (a b)$.
- THESE TWO ARE VERY DIFFERENT!!!! THEY HAVE NOTHING TO DO WITH EACH OTHER!

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- Similarly, $(\exists r_2 \in \mathbb{Z})[a_2 b_2 = m \cdot r_2]$ (II)
- Therefore, by (I) and (II) we have:

$$a_1 - b_1 + a_2 - b_2 = m \cdot r_1 + m \cdot r_2 \Rightarrow (a_1 + a_2) - (b_1 + b_2) = m \cdot (r_1 + r_2) \Rightarrow$$

 $a_1 + a_2 \equiv (b_1 + b_2) (mod \ m)$

2. If $a_1 \equiv b_1 \pmod{m}$ and $a_2 \equiv b_2 \pmod{m}$, then

$$a_1 \cdot a_2 \equiv b_1 \cdot b_2 \pmod{m}$$

Proof: Let $a_1 \equiv b_1 \pmod{m}$ and $a_2 \equiv b_2 \pmod{m}$. By definition, $jm = a_1 - b_1$ and $km = a_2 - b_2$ with $j, k \in \mathbb{Z}$. So, $jm + b_1 = a_1$ and $km + b_2 = a_2$.

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$$a_1 \cdot a_2 = (jm + b_1)(km + b_2)$$

= $jkm^2 + kmb_1 + jmb_2 + b_1 \cdot b_2$
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= $m(jkm + kb_1 + jb_2) + b_1 \cdot b_2$
So, $(a_1 \cdot a_2) - (b_1 \cdot b_2) = m(jkm + kb_1 + jb_2)$. Since
 $jkm + kb_1 + jb_2 \in \mathbb{Z}, a_1 \cdot a_2 \equiv b_1 \cdot b_2 (mod m)$

So, (a₁

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• Using the properties of modular arithmetic, we obtain:

$$a_1 + a_2 \equiv (0+1) \pmod{2} \equiv 1 \pmod{2}$$

• Done.

More proofs

• Similarly, you can show that $(\forall a \in \mathbb{N})[a^2 + a \equiv 0 \pmod{2}]$

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- Similarly, you can show that $(\forall a \in \mathbb{N})[a^2 + a \equiv 0 \pmod{2}]$
- Proof: We will simplify notation by assuming that " \equiv " is the same as

"
$$\equiv (mod \ 2)$$
" We have two cases:
1. $a \equiv 0$. Then, $a^2 + a \equiv 0^2 + 0 \equiv 0$. Done.
2. $a \equiv 1$. Then, $a^2 + a \equiv 1^2 + 1 \equiv 0$. Done.

Algorithms on Divisibility

Modular Exponentiation (Repeated Squaring)
 Greatest Common Divisor (GCD)

Basic assumptions

- a + b and $a \cdot b$ have unit cost
 - This is not true if *a*, *b* are too large

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- Problems
 - Arithmetic overflow in computation of a^n
 - Modding a large quantity is tough on the FPU

First problem, second approach

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- Problems
 - Arithmetic overflow in computation of a^n
 - Modding a large quantity is tough on the FPU
- Additionally, we have another nice property...

• How fast can we compute $a^n \mod m$ $(n, m \in \mathbb{N})$?

We always need *n* steps

We can do it in roughly \sqrt{n} steps

We can do it in roughly logn steps

Something Else

First problem

• How fast can we compute $a^n \mod m \ (n, m \in \mathbb{N})$?



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4. $3^{2^4} \equiv (3^{2^3})^2 \equiv 27^2 \equiv 36$
5. $3^{2^5} \equiv (3^{2^4})^2 \equiv 36^2 \equiv 9$
6. $3^{2^6} \equiv (9)^2 \equiv 81$

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- Aha! $3^{64} = 3^{2^6} \equiv 81$

Good news, bad news

• Good news By using repeated squaring, can compute $a^{2^{\ell}} \mod m$ quickly (roughly $\ell = \log_2 2^{\ell}$ steps)

Good news, bad news

• Good news By using repeated squaring, can compute $a^{2^{\ell}} \mod m$ quickly (roughly $\ell = \log_2 2^{\ell}$ steps)

• Bad news What if our exponent is **not** a power of 2?

• Computing $3^{27} \mod 99$ with the same method

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 - $3^{2^4} \equiv \left(3^{2^3}\right)^2 \equiv 27^2 \equiv 36$
- $3^{27} = 3^{16} \times 3^8 \times 3^2 \times 3^1 \equiv 36 \times 27 \times 9 \times 3$

Example (contd.)

• To avoid large numbers, reduce product as you go

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• $3^{27} = 3^{16} \times 3^8 \times 3^2 \times 3^1 \equiv 36 \times 27 \times 9 \times 3 \equiv$

 $(36 \times 27) \times (9 \times 3) \equiv 81 \times 27 \equiv 9$

Exercise

• Solve the following for *r* please!

 $5^{34} \equiv r \pmod{117}$

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- Step 3 Use repeated squaring to compute

$$a^{2^{0}}, a^{2^{1}}, a^{2^{2}}, \dots, a^{2^{q_{r}}} \mod m$$

using $a^{2^{i+1}} \equiv (a^{2^{i}})^{2} \pmod{m}$

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• Step 4 Compute $a^{2^{q_1}} \times \cdots \times a^{2^{q_r}}$ mod m reducing when necessary to avoid large numbers

• The key step is Step #3. Use repeated squaring to compute

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- When computing $a^{2^{i+1}}$ mod m, already have computed $(a^{2^i})^2 \pmod{m}$
- Note that all numbers are below m because we reduce mod m every step of the way
 - So $(a^{2^i})^2$ is **unit cost** and **anything mod m** is also unit cost!

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- What is the GCD of...
 - 10 and 15?

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 - 12 and 90?

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 - 20 and 29?

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 - 20 and 29? 1 (20 and 29 are called co-prime or relatively prime)
 - 153 and 181

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 - 10 and 15? 5
 - 12 and 90? 6
 - 20 and 29? 1 (20 and 29 are called co-prime or relatively prime)
 - 153 and 181 1 (also co-prime)

Euclid's GCD algorithm

• Recall If $a \equiv 0 \pmod{m}$ and $b \equiv 0 \pmod{m}$, then $a - b \equiv 0 \pmod{m}$

Euclid's GCD algorithm

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$$GCD(a,b) = GCD(a,b - a)$$

Until its arguments are the same.

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• Question If we implement this in a programming language, it can only be done recursively



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$$GCD(a,b) = GCD(a,b - a)$$
 all

Until its arguments are the same.

recursion

while(left != right){
 if(left > right)

else

left = left - right;

right = right - left;

print "GCD is: " left; // Or right

Question If we implement this in a programming language, it can only be done recursively
 left = a; right = b;



GCD example

• GCD(18, 100) =

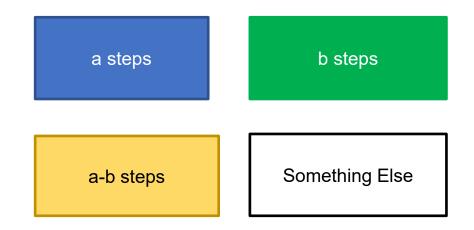
```
GCD(18, 100 - 18) = GCD(18, 82) =
GCD(18, 82 - 18 = GCD(18, 64) =
GCD(18, 64 - 18) = GCD(18, 46) =
GCD(18, 46 - 18) = GCD(18, 28) =
GCD(18, 28 - 18) = GCD(18, 10) =
GCD(18 - 10, 10) = GCD(8, 10) =
GCD(8, 10 - 8) = GCD(8, 2) =
GCD(8 - 2, 2) = GCD(6, 2) =
GCD(6 - 2, 2) = GCD(4, 2) =
GCD(4-2, 2) = GCD(2, 2) = 2
```

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Given integers a, b with a > b (without loss of generality), approximately how many steps does this algorithm take?

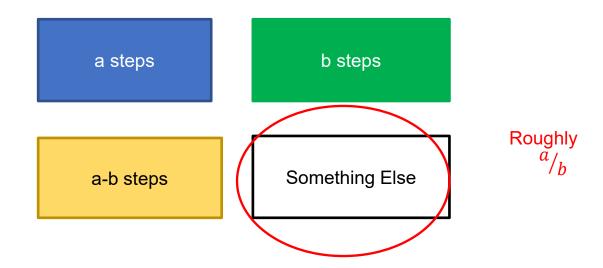


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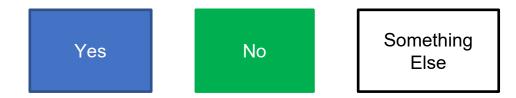
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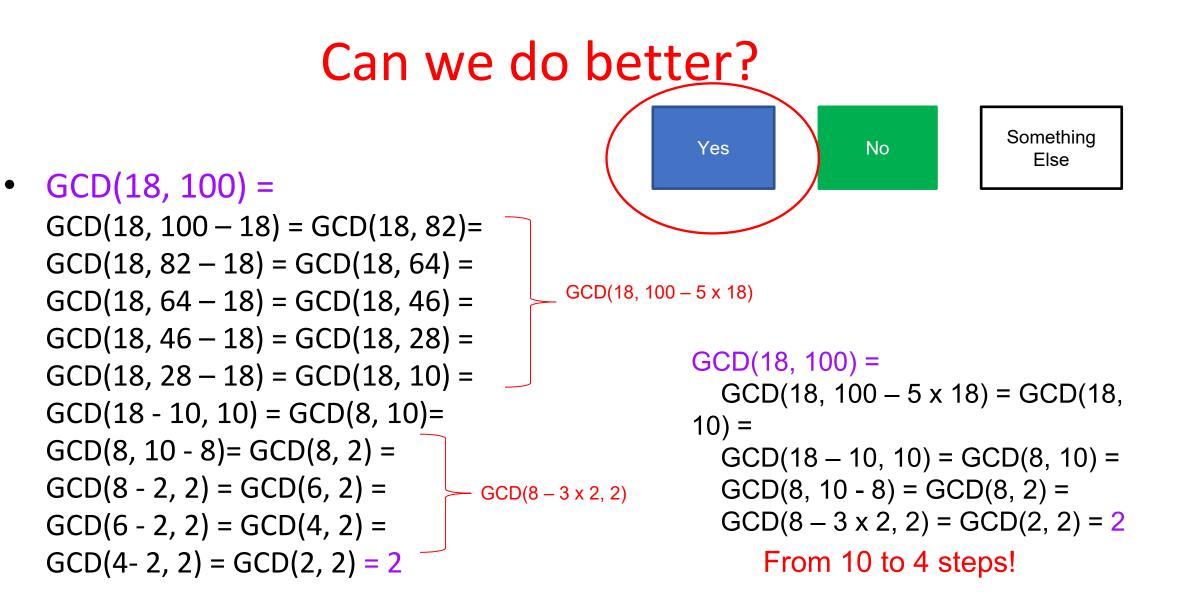


Can we do better?



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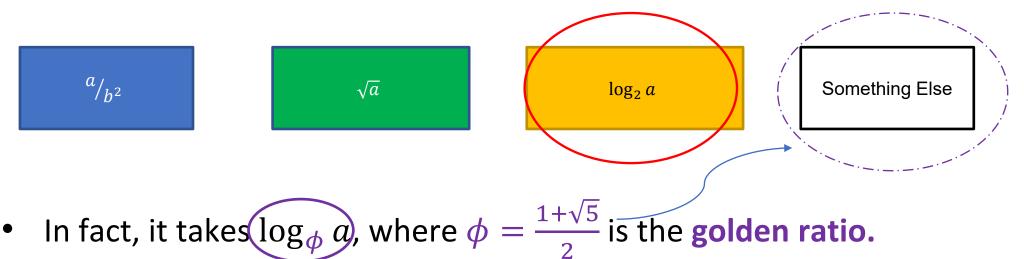
How fast is this new algorithm?

 Given non-zero integers a, b with a > b, roughly how many steps does this new algorithm take to compute GCD(a, b)?



How fast is this new algorithm?

• Given non-zero integers *a*, *b* with *a* > *b*, roughly how many steps does this new algorithm take to compute GCD(a, b)?



 Proof by Gabriel Lamé in 1844, considered by some to be the first ever result in Algorithmic Complexity theory.

ST()P RECORDING