1. (20 points) Give a Propositional Formula on four variables that has exactly three satisfying assignments. Give the satisfying assignments.

**SOLUTION TO PROBLEM ONE**

We give a formula of the form $C_1 \lor C_2 \lor C_3$ so that the three assignments are those that make $C_1$ true, $C_2$ true, $C_3$ true, and make sure that no assignment satisfies two of those.

Vars are $w, x, y, z$

$$(w \land x \land y \land z) \lor (w \land x \land y \land \overline{z}) \lor (w \land x \land \overline{y} \land \overline{z})$$

**EXERCISE FOR YOU:** Do more of these. How many such formulas are there?

**END OF SOLUTION TO PROBLEM ONE**
2. (20 points) Use truth table so show that

\[(x \lor y) \land z\]

is not equivalent to

\[x \lor (y \land z)\].

INDICATE which rows they differ on.

**SOLUTION TO PROBLEM TWO**

Below is the truth table. Here is how I did it with some shortcuts.

Look at the first formula \((x \lor y) \land z\). If \(z = F\) then its false. So I filled in those four entries. For those entries left \(z = T\), so the formula is really \(x \lor y\). So thats \(T\) unless \(x = y = F\).

Look at the second formula \(x \lor (y \land z)\). If \(x\) is true then its true. So I filled in those four entries. For those entries left \(x = F\), so the formula is really \(y \land z\). So thats \(F\) unless \(y = z = T\).

We put a * on the evaluation when the formulas give different values.

<table>
<thead>
<tr>
<th></th>
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<th>((x \lor y) \land z)</th>
<th>(x \lor (y \land z))</th>
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**END OF SOLUTION TO PROBLEM TWO**
3. (30 points) \( n \) has the *emily property* if there is a formula on \( n \) variables with exactly \( n^2 + 100 \) satisfying assignments.

(a) (15 points) Fill in the BLANK in the following sentence

\[
\text{\( \begin{align*}
\text{n has the emily property IFF BLANK(n).} \\
\text{The condition BLANK has to be simple, for example, \( n \) is divisible by 5 (that's not the answer).}
\end{align*}
\) }
\]

(b) (15 points) Prove the statement you made in the first part. Note that this means you have to show that
If BLANK(\( n \)) then \( n \) has the emily property
and
If NOT(BLANK(\( n \))) then \( n \) DOES NOT have the emily property

**SOLUTION TO PROBLEM THREE**

(a) \( n \) has the emily property IFF BLANK(\( n \)).
So we need that there is a boolean formula with exactly \( n^2 + 100 \) satisfying assignments. Here is how you would construct such a formula: Make a Truth Table where the first \( n^2 + 100 \) rows are \( T \) and the rest are \( F \), and then make a formula from that truth table (as shown in class). SO you might thing you can do this for ALL \( n \). But you would be wrong. There are \( 2^n \) rows in a truth table. So we need

\[
n^2 + 100 \leq 2^n
\]
So we need \( 2^n - n^2 - 100 \geq 0 \).
There are two ways to do this:

**METHOD ONE:** Plug \( n = 1, 2, 3, \ldots \) until you get \( 2^n - n^2 - 100 \geq 0 \). Then assume this is true for all larger \( n \) (this is not rigorous, but its true and we’re fine with it).
\( n = 1: 2^1 - 1^2 - 100 < 0 \) so NO
\( n = 2: 2^2 - 2^2 - 100 < 0 \) so NO
WAIT- we need to have \( 2^n \geq 100 \). This might not suffice but we should start there. That's \( n = 7 \) since \( 2^6 = 64 < 100 \) but \( 2^7 = 128 > 100 \).
\( n = 7: \ 2^7 - 7^2 - 100 = 128 - 149 < 0. \)
\( n = 8: \ 2^8 - 8^2 - 100 = 256 - 164 > 0. \) YEAH.
So BLANK\((n)\) is \( n \geq 8. \)

**METHOD TWO:** Let \( f(x) = 2^x - x^2 - 100. \) We need to know when this is always positive. Let’s take the derivative and find max and min
\[
 f'(x) = (\ln 2)2^x - 2x.
\]
One can find that they are two roots, one close to 1 and one close to 3. Evaluating the function in the intervals before and between the roots, one can find out that being 4 the function is increasing.

Now look at the original \( f. \) Its positive for the first time (at an integer) at 8. Since the derivative is positive from 4 on, \( f \) is increasing and hence positive from 8 on.

BLANK\((n)\) is \( n \geq 8. \)

METHOD TWO is messier than METHOD ONE; however, METHOD TWO is more rigorous. If that does not impress you, you are not alone.

(b) I did the prove while doing the problem.

**END OF SOLUTION TO PROBLEM THREE**
4. (30 points) (NOTE: 0 and 1 are NOT prime. You will need that for this problem.)

(a) (15 points) View the input $x, y, z$ as the number in binary $xyz$ which we denote $(xyz)$. For example, 100 is 4.
Write a Truth Table for the following function with 3 inputs $x, y, z$ and 3 outputs $a, b, c$.

\[
f(x, y, z) = \begin{cases} 
0 & \text{if } (xyz) \text{ is NOT PRIME.} \\
1 & \text{if } (xyz) \text{ is PRIME.}
\end{cases}
\]

(b) (15 points) Convert your truth table into formulas. DO NOT SIMPLIFY.

(c) (0 points- DO NOT HAND IN) Draw a circuit that computes that truth table.

**SOLUTION TO PROBLEM FOUR**

(a) Truth Table for IS IT A PRIME

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
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(b) Formula. Look at the rows that evaluate to 1. For each one obtain a mini-fml. Then OR then together.

\[
(\neg a \land b \land \neg c) \lor (\neg a \land \neg b \land c) \lor (a \land \neg b \land c) \lor (a \land b \land c)
\]

**END OF SOLUTION TO PROBLEM FOUR**