Cardinality of Sets

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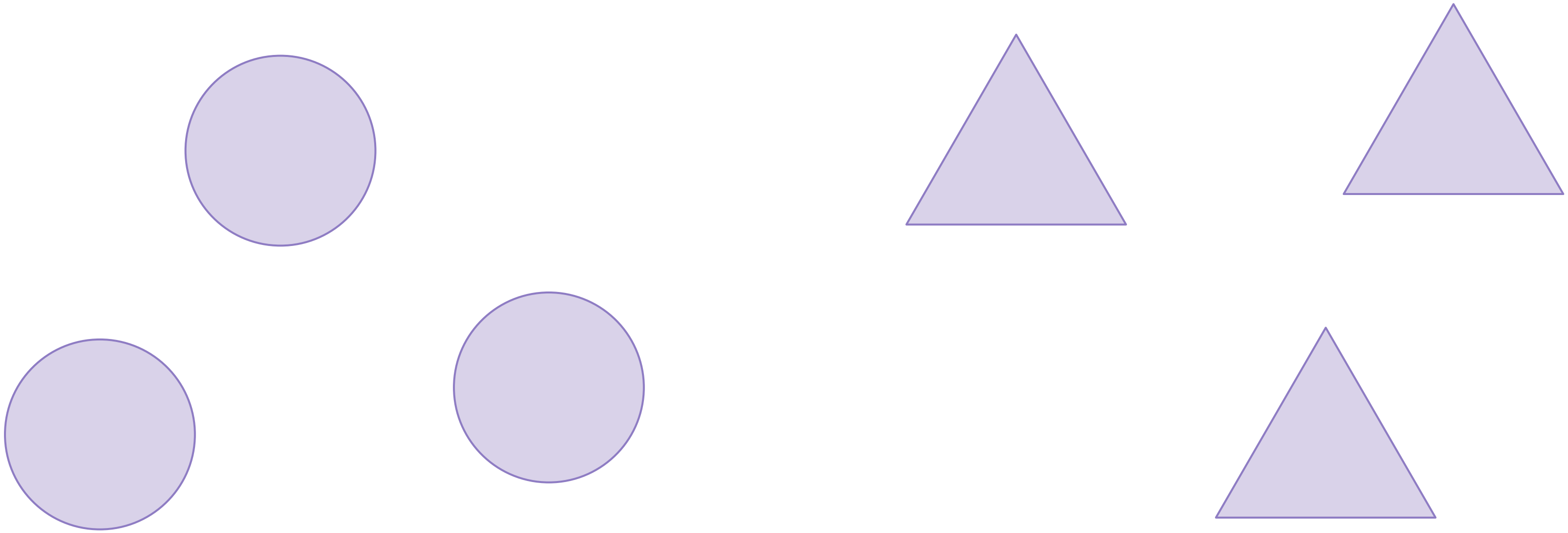
Motivation

- We defined the cardinality of a finite set as the number of elements in the set
- We use the cardinalities of finite sets to tell us when they have the same size, or when one is bigger than the other



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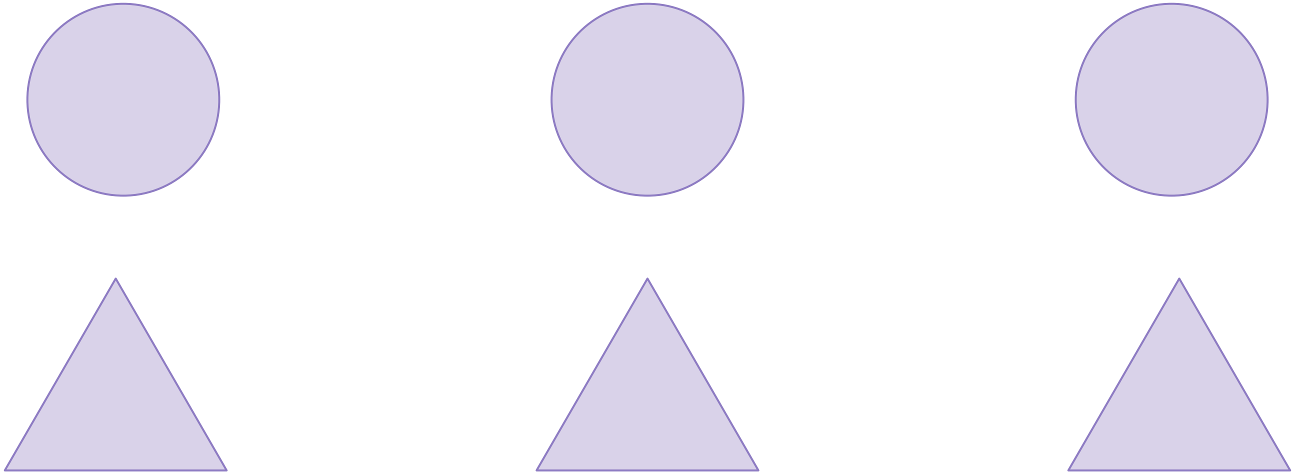

Motivation

- We are now going to extend this notion to infinite sets
  - We will define what it means for two infinite sets to have the same cardinality, providing us with a way to measure the relative sizes of infinite sets.
- We will be particularly interested in ***countably infinite sets***, which are sets with the same cardinality as the set of naturals
- These concepts have important applications to computer science
  - A function is called uncomputable if no computer program can be written to find all its values, even with unlimited time and memory

Cardinality

- Def: ***The sets A and B have the same cardinality if and only if there is a bijection from A to B.*** When A and B have the same cardinality, we write $|A|=|B|$.
- For infinite sets the definition of cardinality provides a relative measure of the sizes of two sets, rather than a measure of the size of one particular set
- Def: If there is a one-to-one function from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A|\leq|B|$. Moreover, when $|A|\leq|B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and we write $|A| < |B|$.
- A set that is either finite or has the same cardinality as the set of naturals it is called ***countable***.
- If A and B are countable sets, then $A \cup B$ is also countable.
- A set that is not countable is called ***uncountable***.
- When an infinite set S is countable, we say it is ***countably infinite***

Countable sets

Is there a bijection between $\mathbf{N}$ and the set of odd positive integers?

Countable sets

Is there a bijection between **N** and the set of odd positive integers?

1	2	3	4	5	6	7	8	9	10	11	12	…
↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	
1	3	5	7	9	11	13	15	17	19	21	23	…

So the odds are a countably infinite set

Is the set of Integers countably infinite? Yes

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	...
0	-1	1	-2	2	-3	3	-4	4	-5	5	-6	6	-7	7	-8	8	-9	9	...

$$f(n) = \begin{cases} 0 & n = 1 \\ -\frac{n}{2} & n = odd \\ \frac{(n-1)}{2} & n = even \end{cases}$$

Is the set of Rationals Countably infinite? Yes

	1	2	3	4	5	6	7	...
1	1/1	2/1	3/1	4/1	5/1	6/1	7/1	...
2	1/2	2/2	3/2	4/2	5/2	6/2	7/2	...
3	1/3	2/3	3/3	4/3	5/3	6/3	7/3	...
4	1/4	2/4	3/4	4/4	5/4	6/4	7/4	...
5	1/5	2/5	3/5	4/5	5/5	6/5	7/5	...
6	1/6	2/6	3/6	4/6	5/6	6/6	7/6	...
7	1/7	2/7	3/7	4/7	5/7	6/7	7/7	...
...	...	...	...	...	...	...	...	...

Is the set of Reals countably infinite? NO

Proof: For the sake of contradiction assume the reals are countable. Then the subset of all real numbers that fall between 0 and 1 are also countable because any subset of a countable set is also countable. Let us list the reals between 0 and 1 in some order.

$$r_1 = 0.d_{11}d_{12}d_{13}d_{14}\ldots$$

$$r_2 = 0.d_{21}d_{22}d_{23}d_{24}\ldots$$

$$r_3 = 0.d_{31}d_{32}d_{33}d_{34}\ldots$$

$$r_4 = 0.d_{41}d_{42}d_{43}d_{44}\ldots$$

$$\vdots$$

where dij ∈ {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}. Let us form a new real number with decimal expansion $r = d_1d_2d_3d_4\ldots$ where we follow this rule

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$$d_i = \begin{cases} 4 & d_{ii} \neq 4 \\ 5 & d_{ii} = 4 \end{cases}$$

Every real number has a unique decimal expansion. Therefore our r is not equal to any of the previous r's as he decimal expansion of differs from the decimal expansion of r_i in the ith place to the right of the decimal point, for each i.

Is the set of Reals countably infinite? NO

Every real number has a unique decimal expansion. Therefore our r is not equal to any of the previous r's as he decimal expansion of differs from the decimal expansion of r_i in the ith place to the right of the decimal point, for each i. Since there is a real number between 0 and 1 that is not in the list we have a contradiction that the numbers between 0 and 1 cannot be listed. So the numbers between 0 and 1 are uncountable and any set with uncountables subsets is uncountable. So, the reals are uncountable. ☽

Proving Infinite Sets

- To prove countably infinite, you would give the bijection
  - OR to prove a set A is countable, you find a set B that we already know is Countable and show $A \subseteq B$.
- To prove uncountability infinite, you would use Cantor's diagonal argument
  - OR to prove a set A is uncountable, you find a set B that we already know is uncountable and show $B \subseteq A$.