Is $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ Dense in $\mathbb{R}$?

Setting

Dense in $\mathbb{R}$

Def Let $\mathbb{D} \subseteq \mathbb{R}$. $\mathbb{D}$ is **dense in** $\mathbb{R}$ if

$$(\forall x, y \in \mathbb{R})[x < y \implies (\exists z \in \mathbb{D})[x < z < y]].$$

Dense in $\mathbb{R}$

Def Let $\mathbb{D} \subseteq \mathbb{R}$. $\mathbb{D}$ is **dense in $\mathbb{R}$** if

$$(\forall x, y \in \mathbb{R})[x < y \implies (\exists z \in \mathbb{D})[x < z < y]].$$

Examples and Counterexamples

Dense in $\mathbb{R}$

Def Let $\mathbb{D} \subseteq \mathbb{R}$. $\mathbb{D}$ is **dense in $\mathbb{R}$** if

$$(\forall x, y \in \mathbb{R})[x < y \implies (\exists z \in \mathbb{D})[x < z < y]].$$

Examples and Counterexamples

1. $\mathbb{Q}$ is dense in $\mathbb{R}$. Just use $\frac{a+b}{2}$.

Dense in $\mathbb{R}$

Def Let $\mathbb{D} \subseteq \mathbb{R}$. $\mathbb{D}$ is **dense in** $\mathbb{R}$ if

$$(\forall x, y \in \mathbb{R})[x < y \implies (\exists z \in \mathbb{D})[x < z < y]].$$

Examples and Counterexamples

1. $\mathbb{Q}$ is dense in $\mathbb{R}$. Just use $\frac{a+b}{2}$.
2. $\mathbb{I}$ (the irrationals) is dense in $\mathbb{R}$. Follows from the untimedmid1 problem.

Dense in $\mathbb{R}$

Def Let $\mathbb{D} \subseteq \mathbb{R}$. $\mathbb{D}$ is **dense in** $\mathbb{R}$ if

$$(\forall x, y \in \mathbb{R})[x < y \implies (\exists z \in \mathbb{D})[x < z < y]].$$

Examples and Counterexamples

1. $\mathbb{Q}$ is dense in $\mathbb{R}$. Just use $\frac{a+b}{2}$.
2. $\mathbb{I}$ (the irrationals) is dense in $\mathbb{R}$. Follows from the untimedmid1 problem.
3. $\mathbb{N}$ and $\mathbb{Z}$ are not dense in $\mathbb{R}$.

More Domains

We will consider the following questions.

More Domains

We will consider the following questions.

1. Is

$$\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$$

   dense in $\mathbb{R}$?

More Domains

We will consider the following questions.

1. Is

$$\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$$

dense in $\mathbb{R}$?
Vote Yes, No, Unknown to Bill.

More Domains

We will consider the following questions.

1. Is

$$\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$$

   dense in $\mathbb{R}$?
   Vote Yes, No, Unknown to Bill.

2. Is

$$\{a + b\pi : a, b \in \mathbb{Z}\}$$

   dense in $\mathbb{R}$?

More Domains

We will consider the following questions.

1. Is

$$\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$$

   dense in $\mathbb{R}$?
   Vote Yes, No, Unknown to Bill.

2. Is

$$\{a + b\pi : a, b \in \mathbb{Z}\}$$

   dense in $\mathbb{R}$?
   Vote Yes, No, Unknown to Bill.

Answer on next slide.

More Domains

More Domains

1. Is

$$\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$$

   dense in $\mathbb{R}$?

More Domains

1. Is

$$\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$$

dense in $\mathbb{R}$? **YES**

More Domains

1. Is

$$\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$$

dense in $\mathbb{R}$? **YES**

2. Is

$$\{a + b\pi : a, b \in \mathbb{Z}\}$$

dense in $\mathbb{R}$?

More Domains

1. Is

$$\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$$

   dense in $\mathbb{R}$? **YES**

2. Is

$$\{a + b\pi : a, b \in \mathbb{Z}\}$$

   dense in $\mathbb{R}$? **YES**

More Domains

1. Is

$$\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$$

   dense in $\mathbb{R}$? **YES**

2. Is

$$\{a + b\pi : a, b \in \mathbb{Z}\}$$

   dense in $\mathbb{R}$? **YES**

3. More generally, if $\gamma \in \mathbb{I}$ then

$$\{a + b\gamma : a, b \in \mathbb{Z}\}$$

   is dense in $\mathbb{R}$

Theorems About $\mathbb{D} = \{a + b\sqrt{2}\}$

We Prove. . . But . . .

We will prove the following:
Thm If $r_1, r_2 \in \mathbb{R}^{>0}$ and $r_1 < r_2$ then

$$(\exists x, y \in \mathbb{Z})[r_1 < x + y\sqrt{2} < r_2].$$

We Prove... But ...

We will prove the following:

Thm If $r_1, r_2 \in \mathbb{R}^{>0}$ and $r_1 < r_2$ then

$$(\exists x, y \in \mathbb{Z})[r_1 < x + y\sqrt{2} < r_2].$$

$\boldsymbol{r_1, r_2 \in \mathbb{R}^{>0}}$ allows us to skip some unimportant details.

We Prove... But ...

We will prove the following:

Thm If $r_1, r_2 \in \mathbb{R}^{>0}$ and $r_1 < r_2$ then

$$(\exists x, y \in \mathbb{Z})[r_1 < x + y\sqrt{2} < r_2].$$

$\boldsymbol{r_1, r_2 \in \mathbb{R}^{>0}}$ allows us to skip some unimportant details.

Using $\sqrt{2}$ instead of $\gamma \in \mathbb{I}$ lets us, as the kids say, **keep it real**.

We Prove. . . But . . .

We will prove the following:
Thm If $r_1, r_2 \in \mathbb{R}^{>0}$ and $r_1 < r_2$ then

$$(\exists x, y \in \mathbb{Z})[r_1 < x + y\sqrt{2} < r_2].$$

$\boldsymbol{r_1, r_2 \in \mathbb{R}^{>0}}$ allows us to skip some unimportant details.

Using $\sqrt{2}$ instead of $\gamma \in \mathbb{I}$ lets us, as the kids say, **keep it real**.

Emily to Bill **Keep it real** is just so 1980's.

We Prove... But ...

We will prove the following:

Thm If $r_1, r_2 \in \mathbb{R}^{>0}$ and $r_1 < r_2$ then

$$(\exists x, y \in \mathbb{Z})[r_1 < x + y\sqrt{2} < r_2].$$

$\boldsymbol{r_1, r_2} \in \mathbb{R}^{>\mathbf{0}}$ allows us to skip some unimportant details.

Using $\sqrt{2}$ instead of $\gamma \in \mathbb{I}$ lets us, as the kids say, **keep it real**.

Emily to Bill **Keep it real** is just so 1980's.

Bill to Emily I'm being retro!

We Prove... But ...

We will prove the following:

Thm If $r_1, r_2 \in \mathbb{R}^{>0}$ and $r_1 < r_2$ then

$$(\exists x, y \in \mathbb{Z})[r_1 < x + y\sqrt{2} < r_2].$$

$\boldsymbol{r_1, r_2 \in \mathbb{R}^{>0}}$ allows us to skip some unimportant details.

Using $\sqrt{2}$ instead of $\gamma \in \mathbb{I}$ lets us, as the kids say, **keep it real**.

Emily to Bill **Keep it real** is just so 1980's.

Bill to Emily I'm being retro! And now, back to the math.

We Prove. . . But . . .

We will prove the following:

Thm If $r_1, r_2 \in \mathbb{R}^{>0}$ and $r_1 < r_2$ then

$$(\exists x, y \in \mathbb{Z})[r_1 < x + y\sqrt{2} < r_2].$$

$\boldsymbol{r_1, r_2 \in \mathbb{R}^{>0}}$ allows us to skip some unimportant details.

Using $\sqrt{2}$ instead of $\gamma \in \mathbb{I}$ lets us, as the kids say, **keep it real**.

Emily to Bill **Keep it real** is just so 1980's.

Bill to Emily I'm being retro! And now, back to the math.

Can use the ideas on these slides to prove:

We Prove... But ...

We will prove the following:

Thm If $r_1, r_2 \in \mathbb{R}^{>0}$ and $r_1 < r_2$ then

$$(\exists x, y \in \mathbb{Z})[r_1 < x + y\sqrt{2} < r_2].$$

$\boldsymbol{r_1, r_2 \in \mathbb{R}^{>0}}$ allows us to skip some unimportant details.

Using $\sqrt{2}$ instead of $\gamma \in \mathbb{I}$ lets us, as the kids say, **keep it real**.

Emily to Bill **Keep it real** is just so 1980's.

Bill to Emily I'm being retro! And now, back to the math.

Can use the ideas on these slides to prove:

Thm Let $\gamma \in \mathbb{I}$. If $r_1, r_2 \in \mathbb{R}$ and $r_1 < r_2$ then

$$(\exists x, y \in \mathbb{Z})[r_1 < x + y\gamma < r_2].$$

Numbers in $\mathbb{D}$ Can Be Small

Thm $(\forall n \in \mathbb{N})(\exists x, y \in \mathbb{Z})\big[0 < x + y\sqrt{2} < \frac{1}{n}\big]$.

Numbers in $\mathbb{D}$ Can Be Small

Thm $(\forall n \in \mathbb{N})(\exists x, y \in \mathbb{Z})\left[0 < x + y\sqrt{2} < \frac{1}{n}\right]$.

Need definitions.

Numbers in $\mathbb{D}$ Can Be Small

Thm $(\forall n \in \mathbb{N})(\exists x, y \in \mathbb{Z})\left[0 < x + y\sqrt{2} < \frac{1}{n}\right]$.

Need definitions.

If $x \in \mathbb{R}^{>0}$ then $\mathbf{H(x)}$ is the part after the decimal point.

Numbers in $\mathbb{D}$ Can Be Small

Thm $(\forall n \in \mathbb{N})(\exists x, y \in \mathbb{Z})\left[0 < x + y\sqrt{2} < \frac{1}{n}\right]$.

Need definitions.

If $x \in \mathbb{R}^{>0}$ then $\mathbf{H}(x)$ is the part after the decimal point.

Example

Numbers in $\mathbb{D}$ Can Be Small

Thm $(\forall n \in \mathbb{N})(\exists x, y \in \mathbb{Z})\left[0 < x + y\sqrt{2} < \frac{1}{n}\right]$.

Need definitions.

If $x \in \mathbb{R}^{>0}$ then $\mathbf{H}(\boldsymbol{x})$ is the part after the decimal point.

Example

$\mathrm{H}(\pi) = 0.14159\ldots$.

Numbers in $\mathbb{D}$ Can Be Small

Thm $(\forall n \in \mathbb{N})(\exists x, y \in \mathbb{Z})\big[0 < x + y\sqrt{2} < \frac{1}{n}\big]$.

Need definitions.

If $x \in \mathbb{R}^{>0}$ then $\mathbf{H}(x)$ is the part after the decimal point.

Example

$\mathrm{H}(\pi) = 0.14159\ldots$.

$\mathrm{H}(\sqrt{3}) = 0.73205\ldots$.

Numbers in $\mathbb{D}$ Can Be Small

Thm $(\forall n \in \mathbb{N})(\exists x, y \in \mathbb{Z})\left[0 < x + y\sqrt{2} < \frac{1}{n}\right]$.

Need definitions.

If $x \in \mathbb{R}^{>0}$ then $\mathbf{H}(\boldsymbol{x})$ is the part after the decimal point.

Example

$\mathrm{H}(\pi) = 0.14159\ldots$.
$\mathrm{H}(\sqrt{3}) = 0.73205\ldots$.
$\mathrm{H}(4) = 0$.

Numbers in $\mathbb{D}$ Can Be Small (cont)

Numbers in $\mathbb{D}$ Can Be Small (cont)

Take the numbers between 0 and 1 and partition them into $\left(0, \frac{1}{n}\right]$, $\left(\frac{1}{n}, \frac{2}{n}\right]$, $\ldots$, $\left(\frac{n-1}{n}, 1\right]$

Numbers in $\mathbb{D}$ Can Be Small (cont)

Take the numbers between 0 and 1 and partition them into $\left(0, \frac{1}{n}\right]$, $\left(\frac{1}{n}, \frac{2}{n}\right]$, ..., $\left(\frac{n-1}{n}, 1\right]$

Map the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$ into those intervals.

Numbers in $\mathbb{D}$ Can Be Small (cont)

Take the numbers between 0 and 1 and partition them into $(0, \frac{1}{n}]$, $(\frac{1}{n}, \frac{2}{n}]$, ..., $(\frac{n-1}{n}, 1]$

Map the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$ into those intervals.

Map (a, b) to the interval that $\mathrm{H}(a + b\sqrt{2})$ is in.

Numbers in $\mathbb{D}$ Can Be Small (cont)

Take the numbers between 0 and 1 and partition them into $\left(0, \frac{1}{n}\right]$, $\left(\frac{1}{n}, \frac{2}{n}\right]$, ..., $\left(\frac{n-1}{n}, 1\right]$

Map the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$ into those intervals.

Map (a, b) to the interval that $\mathrm{H}(a + b\sqrt{2})$ is in.

Example $n = 4$. We map $\{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$ to

Numbers in $\mathbb{D}$ Can Be Small (cont)

Take the numbers between 0 and 1 and partition them into
$\left(0, \frac{1}{n}\right]$, $\left(\frac{1}{n}, \frac{2}{n}\right]$, …, $\left(\frac{n-1}{n}, 1\right]$

Map the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$ into those intervals.

Map (a, b) to the interval that $\mathrm{H}(a + b\sqrt{2})$ is in.

Example $n = 4$. We map $\{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$ to
$\left(0, \frac{1}{4}\right]$, $\left(\frac{1}{4}, \frac{2}{4}\right]$, $\left(\frac{2}{4}, \frac{3}{4}\right]$, $\left(\frac{3}{4}, 1\right]$.

Numbers in $\mathbb{D}$ Can Be Small (cont)

Take the numbers between 0 and 1 and partition them into $\left(0, \frac{1}{n}\right]$, $\left(\frac{1}{n}, \frac{2}{n}\right]$, ..., $\left(\frac{n-1}{n}, 1\right]$

Map the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$ into those intervals.

Map (a, b) to the interval that $\mathrm{H}(a + b\sqrt{2})$ is in.

Example $n = 4$. We map $\{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$ to $\left(0, \frac{1}{4}\right]$, $\left(\frac{1}{4}, \frac{2}{4}\right]$, $\left(\frac{2}{4}, \frac{3}{4}\right]$, $\left(\frac{3}{4}, 1\right]$.
$(0, 0.25]$, $(0.25, 0.5]$, $(0.5, 0.75]$, $(0.75, 1]$.

Numbers in $\mathbb{D}$ Can Be Small (cont)

Take the numbers between 0 and 1 and partition them into
$\left(0, \frac{1}{n}\right]$, $\left(\frac{1}{n}, \frac{2}{n}\right]$, ..., $\left(\frac{n-1}{n}, 1\right]$

Map the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$ into those intervals.

Map (a, b) to the interval that $\mathrm{H}(a + b\sqrt{2})$ is in.

Example $n = 4$. We map $\{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$ to
$\left(0, \frac{1}{4}\right]$, $\left(\frac{1}{4}, \frac{2}{4}\right]$, $\left(\frac{2}{4}, \frac{3}{4}\right]$, $\left(\frac{3}{4}, 1\right]$.
$(0, 0.25]$, $(0.25, 0.5]$, $(0.5, 0.75]$, $(0.75, 1]$.

We show where a few of the ordered pairs go.

Numbers in $\mathbb{D}$ Can Be Small (cont)

Take the numbers between 0 and 1 and partition them into $\left(0, \frac{1}{n}\right], \left(\frac{1}{n}, \frac{2}{n}\right], \ldots, \left(\frac{n-1}{n}, 1\right]$

Map the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$ into those intervals.

Map (a, b) to the interval that $\mathrm{H}(a + b\sqrt{2})$ is in.

Example $n = 4$. We map $\{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$ to
$\left(0, \frac{1}{4}\right], \left(\frac{1}{4}, \frac{2}{4}\right], \left(\frac{2}{4}, \frac{3}{4}\right], \left(\frac{3}{4}, 1\right]$.
$(0, 0.25]$, $(0.25, 0.5]$, $(0.5, 0.75]$, $(0.75, 1]$.

We show where a few of the ordered pairs go.

$(4, 1)$: $4 + 1 \times \sqrt{2} = 5.414$. $\mathrm{H}(4.414) = 0.414 \rightarrow (0.25, 0.5]$.

Numbers in $\mathbb{D}$ Can Be Small (cont)

Take the numbers between 0 and 1 and partition them into $\left(0, \frac{1}{n}\right], \left(\frac{1}{n}, \frac{2}{n}\right], \ldots, \left(\frac{n-1}{n}, 1\right]$

Map the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$ into those intervals.

Map (a, b) to the interval that $\mathrm{H}(a + b\sqrt{2})$ is in.

Example $n = 4$. We map $\{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$ to
$\left(0, \frac{1}{4}\right], \left(\frac{1}{4}, \frac{2}{4}\right], \left(\frac{2}{4}, \frac{3}{4}\right], \left(\frac{3}{4}, 1\right]$.
$(0, 0.25], (0.25, 0.5], (0.5, 0.75], (0.75, 1]$.

We show where a few of the ordered pairs go.

$(4, 1)$: $4 + 1 \times \sqrt{2} = 5.414$. $\mathrm{H}(4.414) = 0.414 \rightarrow (0.25, 0.5]$.
$(3, 2)$: $3 + 2 \times \sqrt{2} = 5.828$. $\mathrm{H}(0.828) = 0.171 \rightarrow (0.75, 1]$.

Numbers in $\mathbb{D}$ Can Be Small (cont)

Take the numbers between 0 and 1 and partition them into
$\left(0, \frac{1}{n}\right], \left(\frac{1}{n}, \frac{2}{n}\right], \ldots, \left(\frac{n-1}{n}, 1\right]$
Map the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$ into those intervals.

Map (a, b) to the interval that $\mathrm{H}(a + b\sqrt{2})$ is in.

Example $n = 4$. We map $\{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$ to
$\left(0, \frac{1}{4}\right], \left(\frac{1}{4}, \frac{2}{4}\right], \left(\frac{2}{4}, \frac{3}{4}\right], \left(\frac{3}{4}, 1\right]$.
$(0, 0.25], (0.25, 0.5], (0.5, 0.75], (0.75, 1]$.

We show where a few of the ordered pairs go.

$(4, 1)$: $4 + 1 \times \sqrt{2} = 5.414$. $\mathrm{H}(4.414) = 0.414 \rightarrow (0.25, 0.5]$.
$(3, 2)$: $3 + 2 \times \sqrt{2} = 5.828$. $\mathrm{H}(0.828) = 0.171 \rightarrow (0.75, 1]$.
$(2, 3)$: $2 + 3 \times \sqrt{2} = 6.242$. $\mathrm{H}(6.242) = 0.242 \rightarrow (0, 0.25]$.

Numbers in $\mathbb{D}$ Can Be Small (cont)

Take the numbers between 0 and 1 and partition them into $\left(0, \frac{1}{n}\right]$, $\left(\frac{1}{n}, \frac{2}{n}\right]$, …, $\left(\frac{n-1}{n}, 1\right]$

Map the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$ into those intervals.

Map (a, b) to the interval that $\mathrm{H}(a + b\sqrt{2})$ is in.

Example $n = 4$. We map $\{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$ to $\left(0, \frac{1}{4}\right]$, $\left(\frac{1}{4}, \frac{2}{4}\right]$, $\left(\frac{2}{4}, \frac{3}{4}\right]$, $\left(\frac{3}{4}, 1\right]$.
$(0, 0.25]$, $(0.25, 0.5]$, $(0.5, 0.75]$, $(0.75, 1]$.

We show where a few of the ordered pairs go.

$(4, 1)$: $4 + 1 \times \sqrt{2} = 5.414$. $\mathrm{H}(4.414) = 0.414 \rightarrow (0.25, 0.5]$.
$(3, 2)$: $3 + 2 \times \sqrt{2} = 5.828$. $\mathrm{H}(0.828) = 0.171 \rightarrow (0.75, 1]$.
$(2, 3)$: $2 + 3 \times \sqrt{2} = 6.242$. $\mathrm{H}(6.242) = 0.242 \rightarrow (0, 0.25]$.
$(1, 4)$: $1 + 4 \times \sqrt{2} = 6.656$. $\mathrm{H}(6.656) = 0.656$. $\rightarrow (0.5, 0.75]$.

Numbers in $\mathbb{D}$ Can Be Small (cont)

In the last slide we described a function from $\{1, \ldots, n\} \times \{1, \ldots, n\}$ to a set of n intervals.

Numbers in $\mathbb{D}$ Can Be Small (cont)

In the last slide we described a function from $\{1, \ldots, n\} \times \{1, \ldots, n\}$ to a set of n intervals.

The domain has n^2 ordered pairs.

Numbers in $\mathbb{D}$ Can Be Small (cont)

In the last slide we described a function from $\{1, \ldots, n\} \times \{1, \ldots, n\}$ to a set of n intervals.

The domain has n^2 ordered pairs.

The co-domain has n intervals.

Numbers in $\mathbb{D}$ Can Be Small (cont)

In the last slide we described a function from $\{1, \ldots, n\} \times \{1, \ldots, n\}$ to a set of n intervals.

The domain has n^2 ordered pairs.

The co-domain has n intervals.

Since $n < n^2$, **by the Pigeonhole Principle** there exists 2 ordered pairs that map to the same interval.
(Actually there exists more but we do not need that.)

Numbers in $\mathbb{D}$ Can Be Small (cont)

In the last slide we described a function from $\{1, \ldots, n\} \times \{1, \ldots, n\}$ to a set of n intervals.

The domain has n^2 ordered pairs.

The co-domain has n intervals.

Since $n < n^2$, **by the Pigeonhole Principle** there exists 2 ordered pairs that map to the same interval.
(Actually there exists more but we do not need that.)

Let (a, b) and (c, d) be two different ordered pairs that map to the same interval.

Numbers in $\mathbb{D}$ Can Be Small (cont)

(a, b) and (c, d) map to the same interval.

Numbers in $\mathbb{D}$ Can Be Small (cont)

(a, b) and (c, d) map to the same interval.

So $\mathrm{H}(a + b\sqrt{2})$ and $\mathrm{H}(c + d\sqrt{2})$ are within $\frac{1}{n}$ of each other.

Numbers in $\mathbb{D}$ Can Be Small (cont)

(a, b) and (c, d) map to the same interval.

So $\mathrm{H}(a + b\sqrt{2})$ and $\mathrm{H}(c + d\sqrt{2})$ are within $\frac{1}{n}$ of each other.

There exists $e, f \in \mathbb{N}$ such that

Numbers in $\mathbb{D}$ Can Be Small (cont)

(a, b) and (c, d) map to the same interval.

So $\mathrm{H}(a + b\sqrt{2})$ and $\mathrm{H}(c + d\sqrt{2})$ are within $\frac{1}{n}$ of each other.

There exists $e, f \in \mathbb{N}$ such that

$\mathrm{H}(a + b\sqrt{2}) = a + b\sqrt{2} - e$

Numbers in $\mathbb{D}$ Can Be Small (cont)

(a, b) and (c, d) map to the same interval.

So $\mathrm{H}(a + b\sqrt{2})$ and $\mathrm{H}(c + d\sqrt{2})$ are within $\frac{1}{n}$ of each other.

There exists $e, f \in \mathbb{N}$ such that

$\mathrm{H}(a + b\sqrt{2}) = a + b\sqrt{2} - e$

$\mathrm{H}(c + d\sqrt{2}) = c + d\sqrt{2} - f$

Numbers in $\mathbb{D}$ Can Be Small (cont)

(a, b) and (c, d) map to the same interval.
So $\mathrm{H}(a + b\sqrt{2})$ and $\mathrm{H}(c + d\sqrt{2})$ are within $\frac{1}{n}$ of each other.
There exists $e, f \in \mathbb{N}$ such that
$\mathrm{H}(a + b\sqrt{2}) = a + b\sqrt{2} - e$
$\mathrm{H}(c + d\sqrt{2}) = c + d\sqrt{2} - f$
We can assume $a + b\sqrt{2} - e < c + d\sqrt{2} - f$.

Numbers in $\mathbb{D}$ Can Be Small (cont)

(a, b) and (c, d) map to the same interval.
So $\mathrm{H}(a + b\sqrt{2})$ and $\mathrm{H}(c + d\sqrt{2})$ are within $\frac{1}{n}$ of each other.
There exists $e, f \in \mathbb{N}$ such that
$\mathrm{H}(a + b\sqrt{2}) = a + b\sqrt{2} - e$
$\mathrm{H}(c + d\sqrt{2}) = c + d\sqrt{2} - f$
We can assume $a + b\sqrt{2} - e < c + d\sqrt{2} - f$.
SO

Numbers in $\mathbb{D}$ Can Be Small (cont)

(a, b) and (c, d) map to the same interval.

So $\mathrm{H}(a + b\sqrt{2})$ and $\mathrm{H}(c + d\sqrt{2})$ are within $\frac{1}{n}$ of each other.
There exists $e, f \in \mathbb{N}$ such that
$\mathrm{H}(a + b\sqrt{2}) = a + b\sqrt{2} - e$
$\mathrm{H}(c + d\sqrt{2}) = c + d\sqrt{2} - f$
We can assume $a + b\sqrt{2} - e < c + d\sqrt{2} - f$.
SO

1. $(c + d\sqrt{2} - f) - (a + b\sqrt{2} - e) < \frac{1}{n}$ since in same interval.

Numbers in $\mathbb{D}$ Can Be Small (cont)

(a, b) and (c, d) map to the same interval.

So $\mathrm{H}(a + b\sqrt{2})$ and $\mathrm{H}(c + d\sqrt{2})$ are within $\frac{1}{n}$ of each other.
There exists $e, f \in \mathbb{N}$ such that
$\mathrm{H}(a + b\sqrt{2}) = a + b\sqrt{2} - e$
$\mathrm{H}(c + d\sqrt{2}) = c + d\sqrt{2} - f$
We can assume $a + b\sqrt{2} - e < c + d\sqrt{2} - f$.
SO

1. $(c + d\sqrt{2} - f) - (a + b\sqrt{2} - e) < \frac{1}{n}$ since in same interval.
2. $(c + d\sqrt{2} - f) - (a + b\sqrt{2} - e) > 0$ since if not then $\sqrt{2} \in \mathbb{Q}$.

Numbers in $\mathbb{D}$ Can Be Small (cont)

(a, b) and (c, d) map to the same interval.

So $\mathrm{H}(a + b\sqrt{2})$ and $\mathrm{H}(c + d\sqrt{2})$ are within $\frac{1}{n}$ of each other.

There exists $e, f \in \mathbb{N}$ such that

$\mathrm{H}(a + b\sqrt{2}) = a + b\sqrt{2} - e$

$\mathrm{H}(c + d\sqrt{2}) = c + d\sqrt{2} - f$

We can assume $a + b\sqrt{2} - e < c + d\sqrt{2} - f$.

SO

1. $(c + d\sqrt{2} - f) - (a + b\sqrt{2} - e) < \frac{1}{n}$ since in same interval.
2. $(c + d\sqrt{2} - f) - (a + b\sqrt{2} - e) > 0$ since if not then $\sqrt{2} \in \mathbb{Q}$.

So

$0 < (c + d\sqrt{2} - f) - (a + b\sqrt{2} - e) < \frac{1}{n}$

$0 < (c + e - f - a) + (b - d)\sqrt{2} < \frac{1}{n}$

Numbers in $\mathbb{D}$ Can Be Small (cont)

(a, b) and (c, d) map to the same interval.
So $\mathrm{H}(a + b\sqrt{2})$ and $\mathrm{H}(c + d\sqrt{2})$ are within $\frac{1}{n}$ of each other.
There exists $e, f \in \mathbb{N}$ such that
$\mathrm{H}(a + b\sqrt{2}) = a + b\sqrt{2} - e$
$\mathrm{H}(c + d\sqrt{2}) = c + d\sqrt{2} - f$
We can assume $a + b\sqrt{2} - e < c + d\sqrt{2} - f$.
SO

1. $(c + d\sqrt{2} - f) - (a + b\sqrt{2} - e) < \frac{1}{n}$ since in same interval.
2. $(c + d\sqrt{2} - f) - (a + b\sqrt{2} - e) > 0$ since if not then $\sqrt{2} \in \mathbb{Q}$.

So
$0 < (c + d\sqrt{2} - f) - (a + b\sqrt{2} - e) < \frac{1}{n}$
$0 < (c + e - f - a) + (b - d)\sqrt{2} < \frac{1}{n}$
So
$(x, y) = (c + e - f - a, b - d) \in \mathbb{Z} \times \mathbb{Z}$ works.

$\mathbb{D}$ is Dense in $\mathbb{R}^{>0}$

Thm If $r_1, r_2 \in \mathbb{R}^{>0}$ and $r_1 < r_2$ then

$$(\exists a, b \in \mathbb{Z})[r_1 < a + b\sqrt{2} < r_2].$$

$\mathbb{D}$ is Dense in $\mathbb{R}^{>0}$

Thm If $r_1, r_2 \in \mathbb{R}^{>0}$ and $r_1 < r_2$ then

$$(\exists a, b \in \mathbb{Z})[r_1 < a + b\sqrt{2} < r_2].$$

Proof

$\mathbb{D}$ is Dense in $\mathbb{R}^{>0}$

Thm If $r_1, r_2 \in \mathbb{R}^{>0}$ and $r_1 < r_2$ then

$$(\exists a, b \in \mathbb{Z})[r_1 < a + b\sqrt{2} < r_2].$$

Proof

Let n be such $\frac{1}{n} < \min\{r_2 - r_1, r_1\}$.

$\mathbb{D}$ is Dense in $\mathbb{R}^{>0}$

Thm If $r_1, r_2 \in \mathbb{R}^{>0}$ and $r_1 < r_2$ then

$$(\exists a, b \in \mathbb{Z})[r_1 < a + b\sqrt{2} < r_2].$$

Proof

Let n be such $\frac{1}{n} < \min\{r_2 - r_1, r_1\}$.

Let $x, y \in \mathbb{Z}$ such that $0 < x + y\sqrt{2} < \frac{1}{n} < \min\{r_2 - r_1, r_1\}$.

$\mathbb{D}$ is Dense in $\mathbb{R}^{>0}$

Thm If $r_1, r_2 \in \mathbb{R}^{>0}$ and $r_1 < r_2$ then

$$(\exists a, b \in \mathbb{Z})[r_1 < a + b\sqrt{2} < r_2].$$

Proof

Let n be such $\frac{1}{n} < \min\{r_2 - r_1, r_1\}$.

Let $x, y \in \mathbb{Z}$ such that $0 < x + y\sqrt{2} < \frac{1}{n} < \min\{r_2 - r_1, r_1\}$.

Continued on next page.

$\mathbb{D}$ is Dense in $\mathbb{R}^{>0}$ (cont)

Want $a, b \in \mathbb{Z}$ such that $r_1 < a + b\sqrt{2} < r_2$.

$\mathbb{D}$ is Dense in $\mathbb{R}^{>0}$ (cont)

Want $a, b \in \mathbb{Z}$ such that $r_1 < a + b\sqrt{2} < r_2$.

Have $x, y \in \mathbb{Z}$ such that $0 < x + y\sqrt{2} < \min\{r_2 - r_1 r_1\}$.

[| ( )

$0 \quad x + y\sqrt{2} \quad r_1 \quad r_2$

$\mathbb{D}$ is Dense in $\mathbb{R}^{>0}$ (cont)

Want $a, b \in \mathbb{Z}$ such that $r_1 < a + b\sqrt{2} < r_2$.

Have $x, y \in \mathbb{Z}$ such that $0 < x + y\sqrt{2} < \min\{r_2 - r_1 r_1\}$.

$[$ $|$ $($ $)$

0 $x + y\sqrt{2}$ r_1 r_2

Consider $2x + 2y\sqrt{2}$, $3x + 3y\sqrt{2}$, $\cdots$

$\mathbb{D}$ is Dense in $\mathbb{R}^{>0}$ (cont)

Want $a, b \in \mathbb{Z}$ such that $r_1 < a + b\sqrt{2} < r_2$.

Have $x, y \in \mathbb{Z}$ such that $0 < x + y\sqrt{2} < \min\{r_2 - r_1 r_1\}$.

[| ( )

0 $x + y\sqrt{2}$ r_1 r_2

Consider $2x + 2y\sqrt{2},\ 3x + 3y\sqrt{2},\ \cdots$

Until you get to an m such that

$\mathbb{D}$ is Dense in $\mathbb{R}^{>0}$ (cont)

Want $a, b \in \mathbb{Z}$ such that $r_1 < a + b\sqrt{2} < r_2$.

Have $x, y \in \mathbb{Z}$ such that $0 < x + y\sqrt{2} < \min\{r_2 - r_1 r_1\}$.

[	|	(	)
0	$x + y\sqrt{2}$	r_1	r_2

Consider $2x + 2y\sqrt{2}$, $3x + 3y\sqrt{2}$, $\cdots$

Until you get to an m such that

$$mx + my\sqrt{2} < r_1 < (m+1)x + (m+1)y\sqrt{2}$$

$\mathbb{D}$ is Dense in $\mathbb{R}^{>0}$ (cont)

Want $a, b \in \mathbb{Z}$ such that $r_1 < a + b\sqrt{2} < r_2$.

Have $x, y \in \mathbb{Z}$ such that $0 < x + y\sqrt{2} < \min\{r_2 - r_1 r_1\}$.

$$\underset{0}{[} \qquad \underset{x+y\sqrt{2}}{|} \qquad\qquad \underset{r_1}{(} \qquad\qquad \underset{r_2}{)}$$

Consider $2x + 2y\sqrt{2}$, $3x + 3y\sqrt{2}$, $\cdots$

Until you get to an m such that

$$mx + my\sqrt{2} < r_1 < (m+1)x + (m+1)y\sqrt{2}$$

Since $x + y\sqrt{2} < r_2 - r_1$ we will have

$$r_1 < (m+1)x + (m+1)y\sqrt{2} < r_2$$

$\mathbb{D}$ is Dense in $\mathbb{R}^{>0}$ (cont)

Want $a, b \in \mathbb{Z}$ such that $r_1 < a + b\sqrt{2} < r_2$.

Have $x, y \in \mathbb{Z}$ such that $0 < x + y\sqrt{2} < \min\{r_2 - r_1 r_1\}$.

[| (   )

0 $x + y\sqrt{2}$ r_1 r_2

Consider $2x + 2y\sqrt{2}$, $3x + 3y\sqrt{2}$, $\cdots$

Until you get to an m such that

$$mx + my\sqrt{2} < r_1 < (m+1)x + (m+1)y\sqrt{2}$$

Since $x + y\sqrt{2} < r_2 - r_1$ we will have

$$r_1 < (m+1)x + (m+1)y\sqrt{2} < r_2$$

$(a, b) = ((m+1)x, (m+1)y)$ works.

Where Did This Come From?

The Origin of the Question

I though of the question

$$\text{is } \{\mathbf{a} + \mathbf{b}\sqrt{\mathbf{2}} : \mathbf{a}, \mathbf{b} \in \mathbb{Z}\} \text{ dense?}$$

The Origin of the Question

I though of the question

$$\text{is } \{\mathbf{a} + \mathbf{b}\sqrt{\mathbf{2}} : \mathbf{a}, \mathbf{b} \in \mathbb{Z}\} \text{ dense?}$$

after the untimed midterm question on density. However, people in math tell me its well known though cannot provide a reference.

The Origin of the Question

I though of the question

$$\text{is } \{\mathbf{a} + \mathbf{b}\sqrt{\mathbf{2}} : \mathbf{a}, \mathbf{b} \in \mathbb{Z}\} \text{ dense?}$$

after the untimed midterm question on density. However, people in math tell me its well known though cannot provide a reference.

What about the proof?

The Origin of the Question

I though of the question

$$\text{is } \{\mathbf{a} + \mathbf{b}\sqrt{\mathbf{2}} : \mathbf{a}, \mathbf{b} \in \mathbb{Z}\} \text{ dense?}$$

after the untimed midterm question on density. However, people in math tell me its well known though cannot provide a reference.

What about the proof?

All of the ideas for the proof were known but in a different context.
It comes from Dirichlets' Theorem on Approximationg Irrationals.
We won't be doing that.

Approximating $\gamma \in \mathbb{I}$ with Rationals

In the early 1800's Dirichlet proved the following:

Approximating $\gamma \in \mathbb{I}$ with Rationals

In the early 1800's Dirichlet proved the following:

Thm Let $\gamma \in \mathbb{I}$. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})[|\gamma - \frac{m}{n}| < \frac{1}{n^2}]$.

Approximating $\gamma \in \mathbb{I}$ with Rationals

In the early 1800's Dirichlet proved the following:
Thm Let $\gamma \in \mathbb{I}$. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})[|\gamma - \frac{m}{n}| < \frac{1}{n^2}]$.
He used the pigeonhole principle.

Approximating $\gamma \in \mathbb{I}$ with Rationals

In the early 1800's Dirichlet proved the following:
Thm Let $\gamma \in \mathbb{I}$. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})[|\gamma - \frac{m}{n}| < \frac{1}{n^2}]$.
He used the pigeonhole principle.

Actually...

Approximating $\gamma \in \mathbb{I}$ with Rationals

In the early 1800's Dirichlet proved the following:
Thm Let $\gamma \in \mathbb{I}$. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})[|\gamma - \frac{m}{n}| < \frac{1}{n^2}]$.
He used the pigeonhole principle.

Actually... It was an early use of the Pigeonhole Principle.

Approximating $\gamma \in \mathbb{I}$ with Rationals

In the early 1800's Dirichlet proved the following:
Thm Let $\gamma \in \mathbb{I}$. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})[|\gamma - \frac{m}{n}| < \frac{1}{n^2}]$.
He used the pigeonhole principle.

Actually... It was an early use of the Pigeonhole Principle.

On Jeopardy: I'll take Historical Math Names for $2000

Approximating $\gamma \in \mathbb{I}$ with Rationals

In the early 1800's Dirichlet proved the following:
Thm Let $\gamma \in \mathbb{I}$. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})[|\gamma - \frac{m}{n}| < \frac{1}{n^2}]$.
He used the pigeonhole principle.

Actually... It was an early use of the Pigeonhole Principle.

On Jeopardy: I'll take Historical Math Names for $2000
Answer An early name for **The Pigeonhole Principle**

Approximating $\gamma \in \mathbb{I}$ with Rationals

In the early 1800's Dirichlet proved the following:
Thm Let $\gamma \in \mathbb{I}$. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})[|\gamma - \frac{m}{n}| < \frac{1}{n^2}]$.
He used the pigeonhole principle.

Actually. . . It was an early use of the Pigeonhole Principle.

On Jeopardy: I'll take Historical Math Names for $2000
Answer An early name for **The Pigeonhole Principle**
Question What is **Dirichlet's Box Principle**?

Approximating $\gamma \in \mathbb{I}$ with Rationals

In the early 1800's Dirichlet proved the following:
Thm Let $\gamma \in \mathbb{I}$. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})[|\gamma - \frac{m}{n}| < \frac{1}{n^2}]$.
He used the pigeonhole principle.

Actually... It was an early use of the Pigeonhole Principle.

On Jeopardy: I'll take Historical Math Names for $2000
Answer An early name for **The Pigeonhole Principle**
Question What is **Dirichlet's Box Principle**?

We prove Dirichlet's Theorem on approximations of irrationals by rationals. You are already familiar with most of the ideas.

Approximating $\gamma \in \mathbb{I}$ with Rationals

Thm Let $\gamma \in \mathbb{I}$. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})\left[|\gamma - \frac{m}{n}| < \frac{1}{n^2}\right]$.

Approximating $\gamma \in \mathbb{I}$ with Rationals

Thm Let $\gamma \in \mathbb{I}$. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})\left[|\gamma - \frac{m}{n}| < \frac{1}{n^2}\right]$.

Take the numbers between 0 and 1 and partition them into $\left(0, \frac{1}{n^2}\right]$, $\left(\frac{1}{n^2}, \frac{2}{n^2}\right]$, …, $\left(\frac{n^2-1}{n^2}, 1\right]$

Approximating $\gamma \in \mathbb{I}$ with Rationals

Thm Let $\gamma \in \mathbb{I}$. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})\left[|\gamma - \frac{m}{n}| < \frac{1}{n^2}\right]$.

Take the numbers between 0 and 1 and partition them into
$\left(0, \frac{1}{n^2}\right]$, $\left(\frac{1}{n^2}, \frac{2}{n^2}\right]$, ..., $\left(\frac{n^2-1}{n^2}, 1\right]$
Map the set $\{1, \ldots, n+1\} \times \{1, \ldots, n+1\}$ into those intervals.

Approximating $\gamma \in \mathbb{I}$ with Rationals

Thm Let $\gamma \in \mathbb{I}$. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})\left[|\gamma - \frac{m}{n}| < \frac{1}{n^2}\right]$.

Take the numbers between 0 and 1 and partition them into
$\left(0, \frac{1}{n^2}\right]$, $\left(\frac{1}{n^2}, \frac{2}{n^2}\right]$, ..., $\left(\frac{n^2-1}{n^2}, 1\right]$

Map the set $\{1, \ldots, n+1\} \times \{1, \ldots, n+1\}$ into those intervals.

Map (a, b) to the interval that $\mathrm{H}(a + b\gamma)$ is in.

Approximating $\gamma \in \mathbb{I}$ with Rationals

In the last slide we described a function from $\{1, \ldots, n+1\} \times \{1, \ldots, n+1\}$ to a set of n^2 intervals.

Approximating $\gamma \in \mathbb{I}$ with Rationals

In the last slide we described a function from $\{1, \ldots, n+1\} \times \{1, \ldots, n+1\}$ to a set of n^2 intervals.

The domain has $(n+1)^2$ ordered pairs.

Approximating $\gamma \in \mathbb{I}$ with Rationals

In the last slide we described a function from $\{1, \ldots, n+1\} \times \{1, \ldots, n+1\}$ to a set of n^2 intervals.

The domain has $(n+1)^2$ ordered pairs.

The co-domain has n^2 intervals.

Approximating $\gamma \in \mathbb{I}$ with Rationals

In the last slide we described a function from $\{1, \ldots, n+1\} \times \{1, \ldots, n+1\}$ to a set of n^2 intervals.

The domain has $(n+1)^2$ ordered pairs.

The co-domain has n^2 intervals.

Since $n^2 < (n+1)^2$, **by the Pigeonhole Principle** there exists 2 ordered pairs that map to the same interval.

Approximating $\gamma \in \mathbb{I}$ with Rationals

In the last slide we described a function from $\{1, \ldots, n+1\} \times \{1, \ldots, n+1\}$ to a set of n^2 intervals.

The domain has $(n+1)^2$ ordered pairs.

The co-domain has n^2 intervals.

Since $n^2 < (n+1)^2$, **by the Pigeonhole Principle** there exists 2 ordered pairs that map to the same interval.

Let (a, b) and (c, d) be two different ordered pairs that map to the same interval.

Approximating $\gamma \in \mathbb{I}$ with Rationals

(a, b) and (c, d) map to the same interval.

Approximating $\gamma \in \mathbb{I}$ with Rationals

(a, b) and (c, d) map to the same interval.

So $\mathrm{H}(a + b\gamma)$ and $\mathrm{H}(c + d\gamma)$ are within $\frac{1}{n^2}$ of each other.

Approximating $\gamma \in \mathbb{I}$ with Rationals

(a, b) and (c, d) map to the same interval.

So $\mathrm{H}(a + b\gamma)$ and $\mathrm{H}(c + d\gamma)$ are within $\frac{1}{n^2}$ of each other.
There exists $e, f \in \mathbb{N}$ such that

Approximating $\gamma \in \mathbb{I}$ with Rationals

(a, b) and (c, d) map to the same interval.

So $\mathrm{H}(a + b\gamma)$ and $\mathrm{H}(c + d\gamma)$ are within $\frac{1}{n^2}$ of each other.
There exists $e, f \in \mathbb{N}$ such that
$\mathrm{H}(a + b\gamma) = a + b\gamma - e$

Approximating $\gamma \in \mathbb{I}$ with Rationals

(a, b) and (c, d) map to the same interval.

So $\mathrm{H}(a + b\gamma)$ and $\mathrm{H}(c + d\gamma)$ are within $\frac{1}{n^2}$ of each other.
There exists $e, f \in \mathbb{N}$ such that
$\mathrm{H}(a + b\gamma) = a + b\gamma - e$
$\mathrm{H}(c + d\gamma) = c + d\gamma - f$

Approximating $\gamma \in \mathbb{I}$ with Rationals

(a, b) and (c, d) map to the same interval.

So $\mathrm{H}(a + b\gamma)$ and $\mathrm{H}(c + d\gamma)$ are within $\frac{1}{n^2}$ of each other.
There exists $e, f \in \mathbb{N}$ such that
$\mathrm{H}(a + b\gamma) = a + b\gamma - e$
$\mathrm{H}(c + d\gamma) = c + d\gamma - f$
We can assume $b - d < 0$.

Approximating $\gamma \in \mathbb{I}$ with Rationals

(a, b) and (c, d) map to the same interval.

So $\mathrm{H}(a + b\gamma)$ and $\mathrm{H}(c + d\gamma)$ are within $\frac{1}{n^2}$ of each other.
There exists $e, f \in \mathbb{N}$ such that
$\mathrm{H}(a + b\gamma) = a + b\gamma - e$
$\mathrm{H}(c + d\gamma) = c + d\gamma - f$
We can assume $b - d < 0$.
SO

Approximating $\gamma \in \mathbb{I}$ with Rationals

(a, b) and (c, d) map to the same interval.

So $\mathrm{H}(a + b\gamma)$ and $\mathrm{H}(c + d\gamma)$ are within $\frac{1}{n^2}$ of each other.
There exists $e, f \in \mathbb{N}$ such that
$\mathrm{H}(a + b\gamma) = a + b\gamma - e$
$\mathrm{H}(c + d\gamma) = c + d\gamma - f$
We can assume $b - d < 0$.
SO

1. $|(c + d\gamma - f) - (a + b\gamma - e)| < \frac{1}{n^2}$ since in same interval.

Approximating $\gamma \in \mathbb{I}$ with Rationals

(a, b) and (c, d) map to the same interval.

So $\mathrm{H}(a + b\gamma)$ and $\mathrm{H}(c + d\gamma)$ are within $\frac{1}{n^2}$ of each other.
There exists $e, f \in \mathbb{N}$ such that
$\mathrm{H}(a + b\gamma) = a + b\gamma - e$
$\mathrm{H}(c + d\gamma) = c + d\gamma - f$
We can assume $b - d < 0$.
SO

1. $|(c + d\gamma - f) - (a + b\gamma - e)| < \frac{1}{n^2}$ since in same interval.
2. $|(c + d\gamma - f) - (a + b\gamma - e)| > 0$ since if not then $\gamma \in \mathbb{Q}$.

Approximating $\gamma \in \mathbb{I}$ with Rationals

(a, b) and (c, d) map to the same interval.

So $\mathrm{H}(a + b\gamma)$ and $\mathrm{H}(c + d\gamma)$ are within $\frac{1}{n^2}$ of each other.
There exists $e, f \in \mathbb{N}$ such that
$\mathrm{H}(a + b\gamma) = a + b\gamma - e$
$\mathrm{H}(c + d\gamma) = c + d\gamma - f$
We can assume $b - d < 0$.
SO

1. $|(c + d\gamma - f) - (a + b\gamma - e)| < \frac{1}{n^2}$ since in same interval.
2. $|(c + d\gamma - f) - (a + b\gamma - e)| > 0$ since if not then $\gamma \in \mathbb{Q}$.

So
$0 < |(c + d\gamma - f) - (a + b\gamma - e)| < \frac{1}{n^2}$
$0 < |(c + a - f - a) + (b - d)\gamma| < \frac{1}{n^2}$
$0 < |(c + a - f - a) - (d - b)\gamma| < \frac{1}{n^2}$

Approximating $\gamma \in \mathbb{I}$ with Rationals

(a, b) and (c, d) map to the same interval.

So $\mathrm{H}(a + b\gamma)$ and $\mathrm{H}(c + d\gamma)$ are within $\frac{1}{n^2}$ of each other.
There exists $e, f \in \mathbb{N}$ such that
$\mathrm{H}(a + b\gamma) = a + b\gamma - e$
$\mathrm{H}(c + d\gamma) = c + d\gamma - f$
We can assume $b - d < 0$.
SO

1. $|(c + d\gamma - f) - (a + b\gamma - e)| < \frac{1}{n^2}$ since in same interval.
2. $|(c + d\gamma - f) - (a + b\gamma - e)| > 0$ since if not then $\gamma \in \mathbb{Q}$.

So
$0 < |(c + d\gamma - f) - (a + b\gamma - e)| < \frac{1}{n^2}$
$0 < |(c + a - f - a) + (b - d)\gamma| < \frac{1}{n^2}$
$0 < |(c + a - f - a) - (d - b)\gamma| < \frac{1}{n^2}$
So
$0 < |\frac{c+a-f-a}{d-b} - \gamma| < \frac{1}{n^2(d-b)} < \frac{1}{n^2}$

Can the Approximation Theorem Be Improved?

Dirichlet Proved:
Thm Let $\gamma \in \mathbb{I}$. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})\left[|\gamma - \frac{m}{n}| < \frac{1}{n^2}\right]$.

Can the Approximation Theorem Be Improved?

Dirichlet Proved:

Thm Let $\gamma \in \mathbb{I}$. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})\left[|\gamma - \frac{m}{n}| < \frac{1}{n^2}\right]$.

Better is known.

Can the Approximation Theorem Be Improved?

Dirichlet Proved:

Thm Let $\gamma \in \mathbb{I}$. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})\left[|\gamma - \frac{m}{n}| < \frac{1}{n^2}\right]$.

Better is known.

Hurwitz proved the following in the late 1800's.

Can the Approximation Theorem Be Improved?

Dirichlet Proved:
Thm Let $\gamma \in \mathbb{I}$. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})\left[|\gamma - \frac{m}{n}| < \frac{1}{n^2}\right]$.
Better is known.
Hurwitz proved the following in the late 1800's.

1. Let $\gamma \in \mathbb{I}$.

$$(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})\left[|\gamma - \frac{m}{n}| < \frac{1}{\sqrt{5}n^2}\right].$$

Can the Approximation Theorem Be Improved?

Dirichlet Proved:
Thm Let $\gamma \in \mathbb{I}$. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})\left[|\gamma - \frac{m}{n}| < \frac{1}{n^2}\right]$.
Better is known.
Hurwitz proved the following in the late 1800's.

1. Let $\gamma \in \mathbb{I}$.

$$(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})\left[|\gamma - \frac{m}{n}| < \frac{1}{\sqrt{5}n^2}\right].$$

   The proof is beyond the scope of this course.

Can the Approximation Theorem Be Improved?

Dirichlet Proved:
Thm Let $\gamma \in \mathbb{I}$. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})\left[|\gamma - \frac{m}{n}| < \frac{1}{n^2}\right]$.

Better is known.
Hurwitz proved the following in the late 1800's.

1. Let $\gamma \in \mathbb{I}$.

$$(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})\left[|\gamma - \frac{m}{n}| < \frac{1}{\sqrt{5}n^2}\right].$$

   The proof is beyond the scope of this course.
2. There exists $\gamma \in \mathbb{I}$ such that, the above is the best possible.

Can the Approximation Theorem Be Improved?

Dirichlet Proved:

Thm Let $\gamma \in \mathbb{I}$. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})\left[|\gamma - \frac{m}{n}| < \frac{1}{n^2}\right]$.

Better is known.

Hurwitz proved the following in the late 1800's.

1. Let $\gamma \in \mathbb{I}$.

$$(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})\left[|\gamma - \frac{m}{n}| < \frac{1}{\sqrt{5}n^2}\right].$$

   The proof is beyond the scope of this course.

2. There exists $\gamma \in \mathbb{I}$ such that, the above is the best possible.
3. $\frac{\sqrt{5}+1}{2}$ is one of those γ.

Questions Left

Questions Left

1. Is the proof that $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is dense easy enough to do in the ordinary CMSC 250?

Questions Left

1. Is the proof that $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is dense easy enough to do in the ordinary CMSC 250?
2. Is there an easier proof that $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is dense?

Questions Left

1. Is the proof that $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is dense easy enough to do in the ordinary CMSC 250?
2. Is there an easier proof that $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is dense? Perhaps one that uses properties of $\sqrt{2}$.

Questions Left

1. Is the proof that $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is dense easy enough to do in the ordinary CMSC 250?
2. Is there an easier proof that $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is dense? Perhaps one that uses properties of $\sqrt{2}$.
3. Is there an easier proof of Hurwitz Theorem?

Questions Left

1. Is the proof that $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is dense easy enough to do in the ordinary CMSC 250?
2. Is there an easier proof that $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is dense? Perhaps one that uses properties of $\sqrt{2}$.
3. Is there an easier proof of Hurwitz Theorem?
4. Has anyone ever wrote this down before?

Questions Left

1. Is the proof that $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is dense easy enough to do in the ordinary CMSC 250?
2. Is there an easier proof that $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is dense? Perhaps one that uses properties of $\sqrt{2}$.
3. Is there an easier proof of Hurwitz Theorem?
4. Has anyone ever wrote this down before?
   This is not an ego-question.

Questions Left

1. Is the proof that $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is dense easy enough to do in the ordinary CMSC 250?
2. Is there an easier proof that $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is dense? Perhaps one that uses properties of $\sqrt{2}$.
3. Is there an easier proof of Hurwitz Theorem?
4. Has anyone ever wrote this down before?
   This is not an ego-question.
   It's part of the more general question of why some people get ideas that others do not.

Questions Left

1. Is the proof that $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is dense easy enough to do in the ordinary CMSC 250?
2. Is there an easier proof that $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is dense? Perhaps one that uses properties of $\sqrt{2}$.
3. Is there an easier proof of Hurwitz Theorem?
4. Has anyone ever wrote this down before?
   This is not an ego-question.
   It's part of the more general question of why some people get ideas that others do not.
   Emily says its because I look at things more pedagogically.