Grid Colorings that Avoid Rectangles

May 3, 2024

Credit Where Credit is Due

This talk is based on a paper by
Stephen Fenner
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Charles Glover
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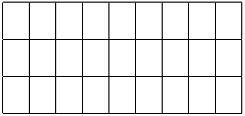
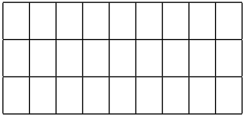
2-Coloring 3×9

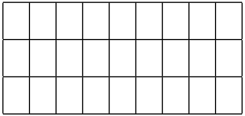


2-Coloring 3×9



Is there a 2-coloring of 3×9 with no mono rectangles?

2-Coloring 3×9



Is there a 2-coloring of 3×9 with no mono rectangles?
What is a mono rectangle? Here is a an example:

2-Coloring 3×9

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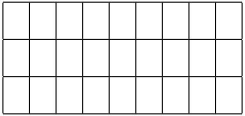
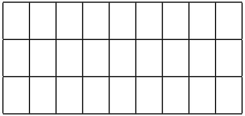
What is a mono rectangle? Here is a an example:

	R					R		
	R					R		

2-Coloring 3×9: Vote



2-Coloring 3×9: Vote



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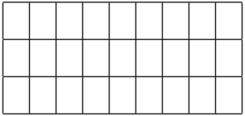

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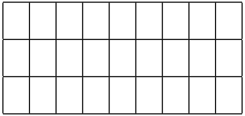
2-Coloring 3×9: Vote



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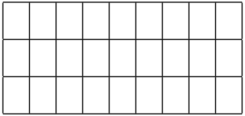
1. There is a 2-coloring of 3×9 with NO mono rectangles.
2. All 2-colorings of 3×9 have a mono rectangle.

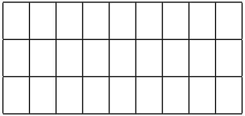
2-Coloring 3×9: Vote



Vote

1. There is a 2-coloring of 3×9 with NO mono rectangles.
2. All 2-colorings of 3×9 have a mono rectangle.
3. The problem is **UNKNOWN TO SCIENCE**.

2-Coloring 3×9: Vote



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Answer on the next slide.

All 2-colorings of 3×9 have a mono rectangle

Given a 2-coloring of 3×9 look at each column.

All 2-colorings of 3×9 have a mono rectangle

Given a 2-coloring of 3×9 look at each column.
A column can either be **RRR** or **RRB** or $\cdots$ or **BBB**.

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Key: A 2-coloring of 3×9 *is an 8-coloring of the 9 columns.*

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So some column-color appears twice.
Example:

	R					R		
	B					B		
	R					R		

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So some column-color appears twice.
Example:

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	R					R		

Can easily show that the two repeat-columns lead to a mono rectangle.

2-Coloring 3×8, 3×7, ...

Work in groups:

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5. Is there a 2-coloring of 3×4 with no mono rectangles?

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4. Is there a 2-coloring of 3×5 with no mono rectangles?
5. Is there a 2-coloring of 3×4 with no mono rectangles?
6. Is there a 2-coloring of 3×3 with no mono rectangles? YES:

Example:

R	B	R
R	B	B
R	R	B

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   NO: to avoid a repeat col must have col **RRR**. Easily get mono rectangle.

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R	R	R	B	B	B
R	B	B	R	B	R
B	R	B	B	R	R

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   YES

R	R	R	B	B	B
R	B	B	R	B	R
B	R	B	B	R	R

4. Is there a 2-coloring of 3×5 with no mono rectangles? YES

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R	R	R	B	B	B
R	B	B	R	B	R
B	R	B	B	R	R

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R	R	R	B	B	B
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4. Is there a 2-coloring of 3×5 with no mono rectangles? YES
5. Is there a 2-coloring of 3×4 with no mono rectangles? YES
6. Is there a 2-coloring of 3×3 with no mono rectangles? YES

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Diff proof that all 2-col of 3×7 have mono rectangle.

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Case 3 ≥ 5 cols have two **R** in them. Map each col to the $\{i,j\}$ such that it has **R** in the ith and jth spot. Domain ≥ 5, range $\binom{3}{2} = 3$ so two cols map to the same $\{i,j\}$. Get mono Rectangle.

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$a \times b$ is *2-colorable* if there is a 2-coloring with no mono rectangles.

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Work on the 4×4, 4×5 4×6.

4×6 IS 2-Colorable

R	R	R	B	B	B
R	B	B	R	R	B
B	R	B	R	B	R
B	B	R	B	R	R

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Work on 5×5, 5×6.

5×5 IS NOT 2-Colorable!

Let COL be a 2-coloring of 5×5.

5×5 IS NOT 2-Colorable!

Let COL be a 2-coloring of 5×5.
Some color must occur ≥ 13 times.

Case 1: There is a column with 5 R's

Case 1: There is a column with 5 **R**'s.

$$\begin{array}{ccccc} \textbf{R} & \circ & \circ & \circ & \circ \\ \textbf{R} & \circ & \circ & \circ & \circ \\ \textbf{R} & \circ & \circ & \circ & \circ \\ \textbf{R} & \circ & \circ & \circ & \circ \\ \textbf{R} & \circ & \circ & \circ & \circ \end{array}$$

Remaining columns have ≤ 1 **R** so

Number of **R**'s $\leq 5 + 1 + 1 + 1 + 1 = 9 < 13.$

Case 2: There is a column with 4 R's

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$$\begin{array}{ccccc} \mathbf{R} & \circ & \circ & \circ & \circ \\ \mathbf{R} & \circ & \circ & \circ & \circ \\ \mathbf{R} & \circ & \circ & \circ & \circ \\ \mathbf{R} & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ \end{array}$$

Remaining columns have ≤ 2 **R**'s

$$\text{Number of } \mathbf{R}\text{'s} \leq 4+2+2+2+2 \leq 12 < 13$$

Case 3: Max in a column is 3 R's

Case 3: Max in a column is 3 **R**'s.
Case 3a: There are ≤ 2 columns with 3 **R**'s.

$$\text{Number of } \mathbf{R}\text{'s} \leq 3+3+2+2+2 \leq 12 < 13.$$

Case 3b: There are ≥ 3 columns with 3 **R**'s.

R	◦	◦	◦	◦
R	◦	◦	◦	◦
R	**R**	◦	◦	◦
◦	**R**	◦	◦	◦
◦	**R**	◦	◦	◦

Can't put in a third column with 3 **R**'s!

Case 4: Max in a column is $\le 2R$'s

Case 4: Max in a column is ≤ 2**R**'s.

$$\text{Number of } \mathbf{R}\text{'s} \le 2+2+2+2+2 \le 10 < 13.$$

No more cases. We are Done! Q.E.D.

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- $3 \times 3, \ldots, 3 \times 6$ 2-colorable.
- $3 \times b$ where $b \geq 7$ NOT 2-colorable.

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We now know *exactly* what grids are 2-colorable.

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We now know *exactly* what grids are 2-colorable.
Can we say it more succinctly?

Obstruction Sets

Def $n \times m$ contains $a \times b$ if $a \leq n$ and $b \leq m$.
Thm For all c there exists a unique finite set of grids OBS_c such that

$n \times m$ is c-colorable **iff**
$n \times m$ does not contain any element of OBS_c.

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1. $\mathrm{OBS}_2 = \{3 \times 7, 7 \times 3, 5 \times 5\}$.
2. Can prove Thm using well-quasi-orderings. No bound on $|\mathrm{OBS}_c|$.
3. We showed $2\sqrt{c}(1 - o(1)) \leq |\mathrm{OBS}_c| \leq 2c^2$.

Main Question

Fix c

What is OBS_c

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Fix c

What is OBS_c

We developed tools to get us both colorings and non-colorings. They helped us get some of our results, but (alas) to many had to be done ad-hoc.

3-COLORABILITY

We will EXACTLY Characterize which $n \times m$ are 3-colorable!

Easy 3-Colorable Results

Thm

1. The following grids **are not** 3-colorable.
   4×19, 19×4, 5×16, 16×5, 7×13, 13×7, 10×12, 12×10, 11×11.
2. The following grids **are** 3-colorable.
   3×19, 19×3, 4×18, 18, 6×15, 15×6, 9×12, 12×9.

Follows from tools.

10×10 is 3-colorable

Thm 10×10 is 3-colorable.
UGLY! TOOLS DID NOT HELP AT ALL!!

R	R	R	R	B	B	G	G	B	G
R	B	B	G	R	R	R	G	G	B
G	R	B	G	R	B	B	R	R	G
G	B	R	B	B	R	G	R	G	R
R	B	G	G	G	B	G	B	R	R
G	R	B	B	G	G	R	B	B	R
B	G	R	B	G	B	R	G	R	B
B	B	G	R	R	G	B	G	B	R
G	G	G	R	B	R	B	B	R	B
B	G	B	R	B	G	R	R	G	G

10×11 is not 3-colorable

Thm 10×11 is not 3-colorable.
You don't want to see this. UGLY case hacking.

Complete Char of 3-colorability

Thm $\mathrm{OBS}_3 =$

$$\{4 \times 19, 5 \times 16, 7 \times 13, 10 \times 11, 11 \times 10, 13 \times 7, 16 \times 5, 19 \times 4\}$$

Follows from our tools and the ad-hoc results.

4-COLORABILITY

From now on $G_{a,b}$ is $a \times b$.

We will EXACTLY Characterize which $G_{n,m}$ are 4-colorable!

Easy NOT 4-Colorable Results

Thm The following grids **are** NOT 4-colorable:

1. $G_{5,41}$ and $G_{41,5}$
2. $G_{6,31}$ and $G_{31,6}$
3. $G_{7,29}$ and $G_{29,7}$
4. $G_{9,25}$ and $G_{25,9}$
5. $G_{10,23}$ and $G_{23,10}$
6. $G_{11,22}$ and $G_{22,11}$
7. $G_{13,21}$ and $G_{21,13}$
8. $G_{17,20}$ and $G_{20,17}$
9. $G_{18,19}$ and $G_{19,18}$

Easy NOT 4-Colorable Results

Thm The following grids **are** NOT 4-colorable:

1. $G_{5,41}$ and $G_{41,5}$
2. $G_{6,31}$ and $G_{31,6}$
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5. $G_{10,23}$ and $G_{23,10}$
6. $G_{11,22}$ and $G_{22,11}$
7. $G_{13,21}$ and $G_{21,13}$
8. $G_{17,20}$ and $G_{20,17}$
9. $G_{18,19}$ and $G_{19,18}$

Follows from our tools.

Easy IS 4-Colorable Results

Thm The following grids **are** 4-colorable:

1. $G_{4,41}$ and $G_{41,4}$.
2. $G_{5,40}$ and $G_{40,5}$.
3. $G_{6,30}$ and $G_{30,6}$.
4. $G_{8,28}$ and $G_{28,8}$.
5. $G_{16,20}$ and $G_{20,16}$.

Easy IS 4-Colorable Results

Thm The following grids **are** 4-colorable:

1. $G_{4,41}$ and $G_{41,4}$.
2. $G_{5,40}$ and $G_{40,5}$.
3. $G_{6,30}$ and $G_{30,6}$.
4. $G_{8,28}$ and $G_{28,8}$.
5. $G_{16,20}$ and $G_{20,16}$.

Follows from our tools.

Theorems with UGLY Proofs

Thm

1. $G_{17,19}$ **is** NOT 4-colorable: Some Tools, Some ad-hoc.
2. $G_{24,9}$ **is** 4-colorable: Some Tools, Some ad-hoc.

Theorems with UGLY Proofs

Thm

1. $G_{17,19}$ **is** NOT 4-colorable: Some Tools, Some ad-hoc.
2. $G_{24,9}$ **is** 4-colorable: Some Tools, Some ad-hoc.

4-coloring of $G_{21,11}$ Due to Brad Loren

	1	2	3	4	5	6	7	8	9	10	11
1	G	B	B	G	R	P	R	G	P	B	P
2	B	G	G	P	B	G	P	R	R	B	R
3	R	R	B	P	B	P	B	P	G	G	R
4	P	R	P	G	B	B	R	P	R	G	B
5	R	P	G	B	B	P	P	B	R	G	G
6	B	R	P	R	G	P	B	R	G	P	B
7	P	G	B	R	G	B	R	G	P	P	R
8	P	P	G	B	R	B	G	R	G	B	P
9	R	B	R	B	G	G	R	P	P	G	B
10	R	P	P	R	G	R	B	B	P	B	G
11	B	P	R	R	P	B	G	G	R	P	G
12	R	B	P	P	P	B	B	R	G	R	G
13	G	G	B	B	R	R	P	P	R	P	G
14	G	B	R	P	B	G	G	R	B	P	P
15	G	P	G	P	G	R	R	R	B	B	B
16	B	B	R	G	P	G	P	B	P	R	G
17	P	G	B	G	P	P	R	B	G	R	B
18	B	P	B	G	G	R	G	P	B	R	R
19	P	G	R	P	R	B	G	B	B	G	R
20	B	R	P	B	R	G	P	G	G	R	P
21	G	R	R	B	P	R	B	P	B	G	P

4-coloring of $G_{22,10}$ Due to Brad Loren

	1	2	3	4	5	6	7	8	9	10
1	P	G	R	R	G	G	P	P	B	B
2	G	P	B	G	B	B	P	R	P	R
3	B	G	B	R	P	P	G	R	P	B
4	P	P	G	G	R	R	B	B	G	P
5	P	B	P	P	G	R	R	G	G	R
6	P	B	R	B	R	P	G	R	G	G
7	G	P	G	P	B	P	R	B	R	G
8	P	R	R	B	P	B	G	G	B	R
9	P	B	B	R	R	G	R	G	P	G
10	R	R	B	B	P	G	R	B	G	P
11	R	G	G	P	R	B	B	G	P	R
12	R	B	R	G	G	P	P	B	B	G
13	B	R	G	B	G	R	B	R	P	P
14	G	G	P	B	B	P	R	R	G	B
15	R	G	P	R	B	R	B	P	P	G
16	B	B	P	G	P	B	P	G	R	R
17	G	P	B	R	P	G	B	P	B	R
18	R	B	G	P	B	G	P	R	R	P
19	G	B	R	P	P	R	B	G	R	B
20	B	R	P	G	R	G	G	B	R	P
21	B	R	G	R	B	P	G	P	B	P
22	G	P	P	R	G	B	G	B	R	B

Absolute Results

Thm

1. The following grids are in OBS_4: $G_{5,41}$, $G_{6,31}$, $G_{7,29}$, $G_{9,25}$, $G_{10,23}$, $G_{11,22}$, $G_{22,11}$, $G_{23,10}$, $G_{25,9}$, $G_{29,7}$, $G_{31,6}$, $G_{41,5}$.
2. For each of the following grids it is not known if it is 4-colorable. These are the only such. $G_{17,17}$, $G_{17,18}$, $G_{18,17}$, $G_{18,18}$. $G_{21,12}$, $G_{22,10}$.
3. Exactly one of these is in OBS_4: $G_{21,11}$, $G_{21,12}$.
4. Exactly one of these is in OBS_4: $G_{17,19}$, $G_{17,18}$, $G_{17,17}$.
5. If $G_{19,17} \in \mathrm{OBS}_4$ then it is possible that $G_{18,18} \in \mathrm{OBS}_4$.

Rectangle Free Conjecture

The following is obvious:
Lemma Let $n, m, c \in \mathbb{N}$. If $G_{n,m}$ is c-colorable then some color occurs $\geq \lceil nm/c \rceil$ times. Hence there is a rectangle free subset of $G_{n,m}$ with $\geq \lceil nm/c \rceil$ elements.

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Rectangle-Free Conjecture (RFC) is the converse:

Let $n, m, c \geq 2$. If there is a rectangle free subset of size of $G_{n,m}$ which is $\geq \lceil nm/c \rceil$ then $G_{n,m}$ is c-colorable.

Rectangle Free Subset of $G_{22,10}$ of Size of size $55 = \left\lceil \frac{22\cdot 10}{4} \right\rceil$

	01	02	03	04	05	06	07	08	09	10
1	•						•			
2		•					•			
3			•				•			
4				•			•			
5					•		•			
6						•	•			
7	•	•						•		
8			•	•				•		
9					•	•		•		
10		•	•						•	
11				•	•				•	
12	•					•			•	
13	•			•						•
14		•				•				•
15			•		•					•
16		•			•					
17	•		•							
18				•		•				
19			•			•				
20		•		•						
21	•				•					
22							•	•	•	•

If RFC is true then $G_{22,10}$ is 4-colorable.

Rectangle Free subset of $G_{21,12}$ of size $63 = \lceil \frac{21 \cdot 12}{4} \rceil$

	01	02	03	04	05	06	07	08	09	10	11	12
1	●	●										
2	●		●									
3		●	●									
4			●	●	●							
5		●		●		●						
6	●				●	●						
7						●	●	●				
8					●		●		●			
9				●				●	●			
10						●				●	●	
11					●					●		●
12				●							●	●
13			●			●			●			●
14			●					●		●		
15			●				●				●	
16		●							●	●		
17		●			●			●			●	
18		●					●					●
19	●								●		●	
20	●							●				●
21	●			●			●			●		

If RFC is true then $G_{21,12}$ is 4-colorable.

Rectangle Free subset of $G_{18,18}$ of size $81 = \lceil \frac{18 \cdot 18}{4} \rceil$

	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18
1		●		●										●		●	●	
2	●	●								●	●		●					
3	●								●						●	●		●
4						●			●			●	●	●				
5		●	●			●												●
6	●			●		●	●											
7							●	●		●				●				●
8			●				●		●		●						●	
9		●			●		●					●			●			
10				●							●	●						●
11	●		●		●									●				
12			●	●				●					●		●			
13					●	●		●			●					●		
14	●							●				●					●	
15				●	●				●	●								
16						●				●					●		●	
17			●							●		●				●		
18					●								●				●	●

If RFC is true then $G_{18,18}$ is 4-colorable. NOTE: If delete 2nd column and 5th Row have 74-sized RFC of $G_{17,17}$.

Assuming RFC...

Thm If RFC then

$$\mathrm{OBS}_4 = \{G_{41,5}, G_{31,6}, G_{29,7}, G_{25,9}, G_{23,10}, G_{22,11}, G_{21,13}, G_{19,17}\} \bigcup$$

$$\{G_{13,21}, G_{11,22}, G_{10,23}, G_{9,25}, G_{7,29}, G_{6,31}, G_{5,41}\}.$$

Follows from known 4-colorability, non-4-colorability results, and Rect Free Sets above.

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So OBS_4 is known!

OPEN QUESTIONS

1. What is OBS_5?
2. Prove or disprove **Rectangle Free Conjecture.**
3. Have $\Omega(\sqrt{c}) \le |\mathrm{OBS}_c| \le O(c^2)$. Get better bounds!
4. Refine tools so can prove **ugly** results **cleanly**.