3-Colorings of $4 \times x$ that Avoid Rectangles

May 1, 2024

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For all 3-colorings of $4 \times x$ there exists a mono rectangle.

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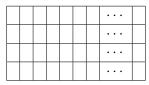
Try to make x small

Bonus if $4 \times (x - 1)$ is 3-colorable.

This is modeled after our proof that 3×9 is not 2-colorable..

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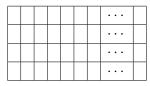
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Given a 3-coloring of $4 \times x$ look at each column.

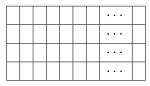
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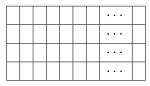
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Given a 3-coloring of $4 \times x$ look at each column. Column is **RRRR** or \cdots or **GGGG**.Possibilities: $3 \times 3 \times 3 \times 3 = 81$.

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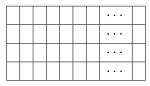
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Key: A 3-coloring of $4 \times x$ is an 81-coloring of the x columns.

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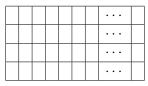


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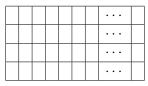
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If x = 82 then some column-color appears twice. Example:

R		R	
B		В	
R		R	
G		G	

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Can show that the 2 repeat-columns imply a mono rectangle,

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Map every column to one of the colors that appears twice and the pair of spots its at.

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New Approach $4 \times x$ is 3-colored.

Every column has two of the same color (perhaps more).

Map every column to one of the colors that appears twice and the pair of spots its at. Example on next page.



1st column could map to either $R \times \{1,2\}$ or $B \times \{3,4\}$.





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4th col maps to $B \times \{1,3\}$.

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2nd col maps to $R \times \{1, 4\}$.

3rd col maps to $G \times \{1,2\}$.

4th col maps to $B \times \{1, 3\}$.

5th col maps to $B \times \{1, 2\}$.

6th col maps to $R \times \{2,3\}$.

We are mapping every column to a pair:



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We are mapping every column to a pair:

- ► A color. There are 3 of those.
- An unordered pair from $\{1, 2, 3, 4\}$. $\binom{4}{2} = 6$ of those.
- Hence there are $3 \times 6 = 18$ in the range.
- We can take x = 19 to guarantee two columns map to the same color and spots.

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Is x = 19 optimal?

Every 3-coloring of 4 \times 19 has a mono rectangle.



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Is x = 19 optimal?

Every 3-coloring of 4×19 has a mono rectangle. Is there a 3-coloring of 4×19 ? Yes. On next slide.

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3-coloring of 4 \times 18

R	R	R	В	В	В
R	В	В	R	R	G
В	R	G	R	G	R
G	G	R	G	R	R

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3-coloring of 4 \times 18

R	R	R	В	В	В
R	В	В	R	R	G
В	R	G	R	G	R
G	G	R	G	R	R

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This is a coloring of 4×6 .

3-coloring of 4×18

R	R	R	В	В	В
R	В	В	R	R	G
В	R	G	R	G	R
G	G	R	G	R	R

This is a coloring of 4×6 .

Rotate **R** to **B**, **B** to **G** and **G** to **R** and that gives 6 more column. Coloring of 4×6 where every column has 2 **B**'s, 1 **R**, 1 **G**.

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3-coloring of 4 \times 18

R	R	R	В	В	В
R	В	В	R	R	G
В	R	G	R	G	R
G	G	R	G	R	R

This is a coloring of 4×6 .

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3-coloring of 4 imes 18

R	R	R	В	В	В
R	В	В	R	R	G
В	R	G	R	G	R
G	G	R	G	R	R

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Rotate **R** to **B**, **B** to **G** and **G** to **R** and that gives 6 more column. Coloring of 4×6 where every column has 2 **G**'s, 1 **B**, 1 **R**.

Put these colorings side by side to get 3-coloring of 4×18 .