3-Colorings of $4 \times x$ that Avoid Rectangles

May 1, 2024

Goal: Find an x such that ...

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For all 3-colorings of $4 \times x$ there exists a mono rectangle.

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Bonus if $4 \times (x-1)$ is 3-colorable.

First Answer

This is modeled after our proof that 3×9 is not 2-colorable..

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							$\cdots$	
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Given a 3-coloring of $4 \times x$ look at each column.

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Column is **RRRR** or $\cdots$ or **GGGG**.

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Column is **RRRR** or $\cdots$ or **GGGG**.Possibilities: $3 \times 3 \times 3 \times 3 = 81$.

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Key: A 3-coloring of $4 \times x$ *is an 81-coloring of the* x *columns.*

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If $x = 82$ then some column-color appears twice.

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	R					**R**		
	B					**B**		
	R					**R**		
	G					**G**		

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Can show that the 2 repeat-columns imply a mono rectangle.

Can we do Better than 82?. Yes!

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Map every column to one of the colors that appears twice and the pair of spots its at.

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Map every column to one of the colors that appears twice and the pair of spots its at.
Example on next page.

Example of the Map

R	R	G	B	B	B
R	B	G	G	B	R
B	G	G	B	G	R
B	R	G	B	R	G

1st column could map to either $R \times \{1, 2\}$ or $B \times \{3, 4\}$.

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R	R	G	B	B	B
R	B	G	G	B	R
B	G	G	B	G	R
B	R	G	B	R	G

1st column could map to either $R \times \{1, 2\}$ or $B \times \{3, 4\}$.
We take the pair of numbers with the smallest first elt, so $R \times \{1, 2\}$.

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R	B	G	G	B	R
B	G	G	B	G	R
B	R	G	B	R	G

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2nd col maps to $R \times \{1,4\}$.

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4th col maps to $B \times \{1, 3\}$.

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4th col maps to $B \times \{1, 3\}$.

5th col maps to $B \times \{1, 2\}$.

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3rd col maps to $G \times \{1, 2\}$.

4th col maps to $B \times \{1, 3\}$.

5th col maps to $B \times \{1, 2\}$.

6th col maps to $R \times \{2, 3\}$.

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We are mapping every column to a pair:

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We are mapping every column to a pair:

- A color. There are 3 of those.
- An unordered pair from $\{1, 2, 3, 4\}$. $\binom{4}{2} = 6$ of those.
- Hence there are $3 \times 6 = 18$ in the range.
- We can take $x = 19$ to guarantee two columns map to the same color and spots.

Is $x = 19$ optimal?

Every 3-coloring of 4×19 has a mono rectangle.

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Is there a 3-coloring of 4×19?

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Is there a 3-coloring of 4×19?
Yes. On next slide.

3-coloring of 4×18

R	R	R	B	B	B
R	B	B	R	R	G
B	R	G	R	G	R
G	G	R	G	R	R

3-coloring of 4×18

R	R	R	B	B	B
R	B	B	R	R	G
B	R	G	R	G	R
G	G	R	G	R	R

This is a coloring of 4×6.

3-coloring of 4×18

R	R	R	B	B	B
R	B	B	R	R	G
B	R	G	R	G	R
G	G	R	G	R	R

This is a coloring of 4×6.

Rotate **R** to **B**, **B** to **G** and **G** to **R** and that gives 6 more column.
Coloring of 4×6 where every column has 2 **B**'s, 1 **R**, 1 **G**.

3-coloring of 4 × 18

R	R	R	B	B	B
R	B	B	R	R	G
B	R	G	R	G	R
G	G	R	G	R	R

This is a coloring of 4×6.

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Rotate **R** to **B**, **B** to **G** and **G** to **R** and that gives 6 more column. Coloring of 4×6 where every column has 2 **G**'s, 1 **B**, 1 **R**.

3-coloring of 4×18

R	**R**	**R**	**B**	**B**	**B**
R	**B**	**B**	**R**	**R**	**G**
B	**R**	**G**	**R**	**G**	**R**
G	**G**	**R**	**G**	**R**	**R**

This is a coloring of 4×6.

Rotate **R** to **B**, **B** to **G** and **G** to **R** and that gives 6 more column. Coloring of 4×6 where every column has 2 **B**'s, 1 **R**, 1 **G**.

Rotate **R** to **B**, **B** to **G** and **G** to **R** and that gives 6 more column. Coloring of 4×6 where every column has 2 **G**'s, 1 **B**, 1 **R**.

Put these colorings side by side to get 3-coloring of 4×18.