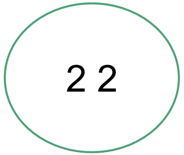
Nim Games

250H

How to Play

- To players take turns removing objects from distinct piles
  - You can have any number of piles and any amount of objects in each pile
- Each player must remove at least 1 object and may remove any number of objects as long as they all come from the same pile
- Depending on the version: the goal of the game is either to
  - Avoid taking the last object
  - To take the last object

We can use game trees to look at all possible games (the players are playing perfectly here)



Player 1’s turn

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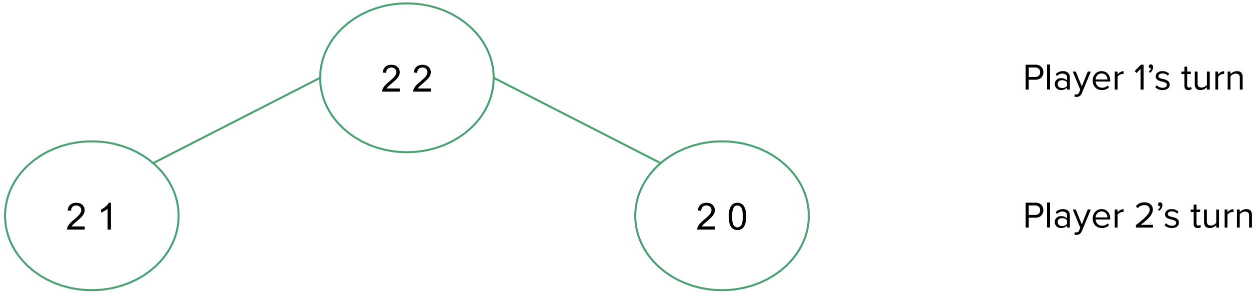

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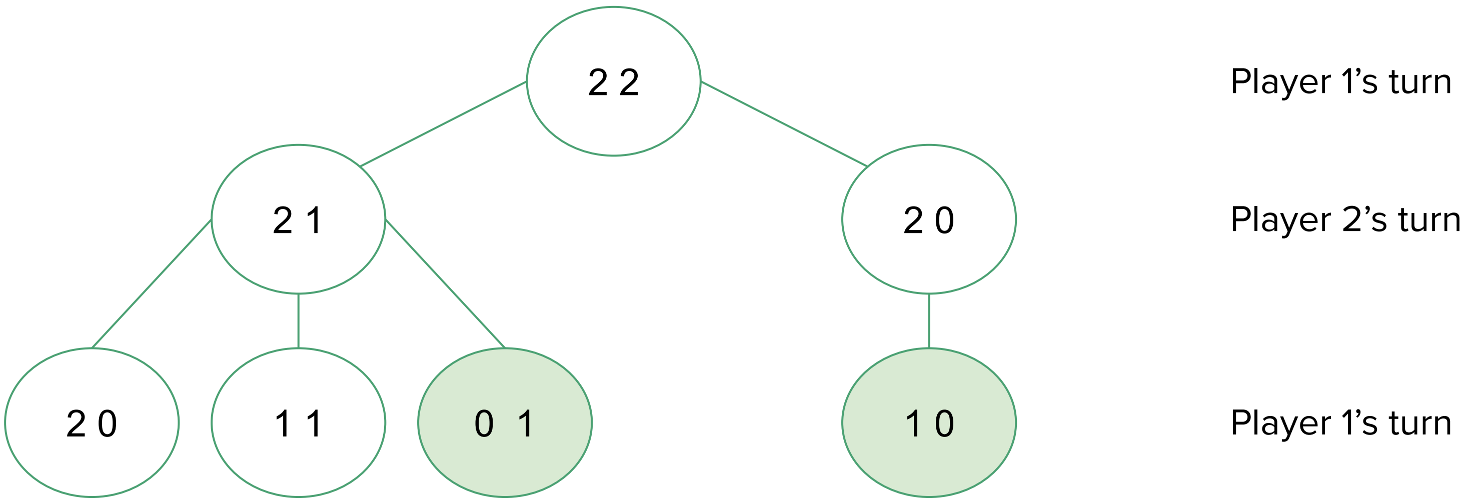

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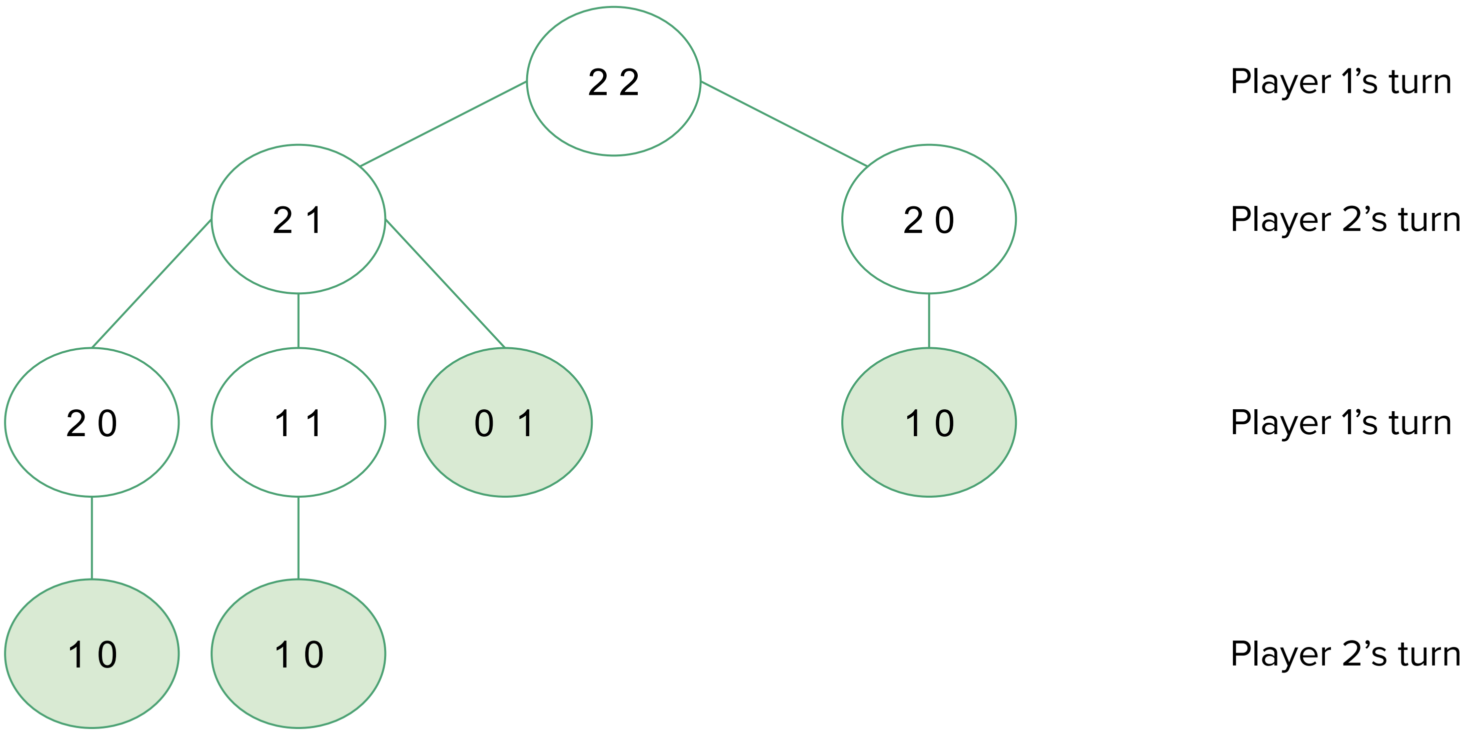

Consider a 2 pile game of Nim where you win if you pick up the last stone. Prove if both piles of stones have n stones each and it's the first player's turn, the second player can always win.

Base Case: If both piles have 0 stones in them, the first player loses

Inductive Hypothesis: Assume that for some $n \geq 0$ and $0 \leq i < n$. If both piles have i number of stones and it's the first player's turn, the second player can win.

Inductive Step: Consider a game of Nim in which there are two piles of stones, A and B, with n stones in each. Without loss of generality, let A be the pile that the first player chooses to remove stones from.

The first player must remove k stones from pile A such that $1 \leq k \leq n$. So, we have $n - k$ stones in pile A and n stones in pile B.

If the second player removes k stones from pile B, both piles have $n - k$ stones in each.

By the induction hypothesis, the second player can now win this game because there are two piles with $n - k$ stones in each.

What is the winning strategy?

- Need to write the sizes of the piles in binary
- Add those numbers up but not in the usual way (AKA use XOR)
    - If the number of 1's in a column is odd, you write a 1 underneath it
    - If it's even, you write a 0 underneath it.
    - Doing this for each column gives a new binary number, and that's the result of the Nim addition.

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- Example:
  - Pile 1 has 2 objects
  - Pile 2 has 3 objects
  - $$\begin{array}{r} 10 \\ +\,11 \\ \hline 01 \end{array}$$

What is the winning strategy?

- Charles Bouton studied this game and figured out two things
  - Suppose it's your turn and the Nim sum of the number of objects in the pile is equal to 0
    - The Nim sum of the number of objects after your move will not be equal to 0
  - Suppose it's your turn and the Nim sum of the number of objects in the pile is not equal to 0
    - Then there is a move which ensures that the Nim sum of the number of objects in the pile after your move is equal to 0

What is the winning strategy?

- Let player 1 go first and the Nim sum of the number of objects in the piles not be equal to 0
- Player 1’s strategy: if possible always make a move that reduces the Nim sum after your move to 0
- This would then mean that whatever player 2 does next, the move would turn the next Nim sum into a number that's not 0
- Player 1 wins IFF there is a move he can make that puts the game into a Player 2 win position

Variant: You have 1 pile. Players can only remove a square number of objects. The player who removes the last object wins

- What is the winning strategy?
  - Let 0 be bad and 1 be good
  - If all numbers 1 .. N have been labeled as either bad or good, then
    - The number N+1 is bad if only good numbers can be reached by subtracting a positive square
    - The number N+1 is good if at least one bad number can be reached by subtracting a positive square
  - The winning strategy of the game: Try to pass on a bad number to your opponent

Variant: You have 1 pile. Players can only remove 1, 2, or 3 objects. The player who removes the last object wins

- What is the winning strategy?
  - If there are only 1, 2, or 3 objects left on your turn, you take all of them
  - If you have to move when there are 4 objects you will always lose
    - No matter how many you take, you will leave 1, 2, or 3
  - If there are 5, 6, or 7 objects, you can win by taking just enough to leave 4 objects
  - The winning strategy of the game: At the end of your turn, make it so that your opponent is taking from a multiple of 4 objects

Variant: You have 1 pile. Players can only remove 1, 3, or 4 objects. The player who removes the last object wins

- What is the winning strategy?
  - If there are only 1, 3, or 4 objects left on your turn, you take all of them
  - If you have to move when there are 2 objects you will always lose
    - You will leave 1
  - If there are 5, you can win by taking 3 objects
  - If there are 6, you can win by taking 4 objects
  - If you have to move when there are 7 objects you will always lose
  - The winning strategy of the game: At the end of your turn, make it so that your opponent is taking from a pile that is equivalent to 2 or 0 mod 7

Variant: You have 2 piles. Players can remove as many as they want from either OR the SAME amount from both. A player wins when they remove the last object.

- What is the winning strategy?
  - Any position in the game can be described by a pair of integers (n, m) with $n \leq m$, where n and m are the piles
  - The strategy of the game revolves around cold positions and hot positions:
    - Cold Position: the player whose turn it is to move will lose when playing perfectly
    - Hot Position: the player whose turn it is to move will win when playing perfectly
    - The optimal strategy from a hot position is to move to any cold position
  - The classification of positions into hot and cold can be looked at recursively:
    - (0,0) is a cold position
    - Any position from which a cold position can be reached in a single move is a hot position
    - If every move leads to a hot position, then a position is cold.