Duplicator Spoiler Games Revisited

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Our Interest Given L, L' find the smallest k such that S wins.

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If $Q \in \{\exists, \forall\}$ then

$$\mathrm{qd}((Qx_1)[\phi(x_1, \ldots, x_n)] = \mathrm{qd}(\phi_1(x_1, \ldots, x_n)) + 1.$$

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$$\mathrm{qd}((\forall x)(\forall z)[x < z \rightarrow (\exists y)[x < y < z]]) = 2 + 1 = 3$$

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Def L and L' are *k-truth-equiv* ($L \equiv_k^T L'$)

$$(\forall \phi, qd(\phi) \le k)[L \models \phi \text{ iff } L' \models \phi.]$$

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3. Upshot: Questions about expressability become questions about games.
4. Complexity: As Computer Scientists we think of complexity in terms of time or space (e.g., sorting n elements can be done in roughly $n \log n$ comparisons). But how do you measure complexity for concepts where time and space do not apply? One measure is quantifier depth. These games help us prove LOWER BOUNDS on quantifier depth!

Proving DUP Wins Rigorously

Notation

The game where the orders are L and L', and its for n moves, will be denoted

$$(L, L'; n)$$

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  Since $x \leq 2^{n-1}$ and $a, b \geq 2^n$, $a - x - 1 \geq 2^{n-1}$ and $b - x - 1 \geq 2^{n-1}$.
  By IH DUP wins $(L_{a-x}, L_{b-x}; n-1)$.

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3. Might not need induction on the smaller boards if they are orderings we already proved things about.

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$(\mathbb{N}+\mathbb{N}^*,L_{2^n};n-1)$ and $(\mathbb{N},\mathbb{N};n-1)$.
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I leave the rest to you