More Induction Problems CMSC 250

1. Prove $21 \mid (4^{n+1} + 5^{2n-1})$ for every positive integer n.

   Proof:
   Base Case: Let $n = 1$. So, $4^{n+1}+5^{2n-1} = 4^{1+1}+5^{2(1)-1} = 16+5 = 21$. Since $21 \mid 21$, our base holds.
   Inductive Hypothesis: Assume for some integer positive integer k, $21 \mid (4^{k+1} + 5^{2k-1})$.
   Inductive Step: Consider $n = k + 1$. So,

   $$4^{k+1+1} + 5^{2(k+1)-1}$$

   $$4^{k+2} + 5^{2k+1}$$

   $$(4)4^{k+1} + 5^2(5^{2k-1})$$

   $$(4)4^{k+1} + 25(5^{2k-1})$$

   $$(4)4^{k+1} + (21 + 4)(5^{2k-1})$$

   $$(4)4^{k+1} + 21(5^{2k-1}) + 4(5^{2k-1})$$

   $$4(4^{k+1} + 5^{2k-1}) + 21(5^{2k-1})$$

   From our inductive hypothesis, we know $21 \mid 4^{k+1}+5^{2k-1}$. Since $21 \mid 21$, $21 \mid (4(4^{k+1} + 5^{2k-1}) + 21(5^{2k-1}))$. Therefore by PMI, our statement holds𝔇

2. Prove that for every positive integer n,

   $$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1)$$

   Proof:
   Base Case: Let $n = 1$. Then,

   $$2(\sqrt{1+1} - 1)$$

   $$= 2(\sqrt{2} - 1)$$

   $$\approx 0.828$$

Since $1 > 0.828$, our base case holds.
Inductive Hypothesis: Assume for some integer $k \geq 1$,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{k}} > 2(\sqrt{k+1} - 1)$$

Inductive Step: Let $n = k + 1$. So,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$$

From our inductive hypothesis, we have

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > 2(\sqrt{k+1} - 1) + \frac{1}{\sqrt{k+1}}$$

Note that we need to show

$$2(\sqrt{k+1} - 1) + \frac{1}{\sqrt{k+1}} > 2(\sqrt{(k+2)} - 1)$$

So,

$$\frac{1}{\sqrt{k+1}} > 2(\sqrt{(k+2)} - 1) - 2(\sqrt{k+1} - 1)$$

$$\frac{1}{\sqrt{k+1}} > 2(\sqrt{(k+2)} - \sqrt{k+1})$$

Note that we can turn

$$2(\sqrt{(k+2)} - \sqrt{k+1})$$

into

$$2(\sqrt{k+2} - \sqrt{k+1})(\sqrt{k+2} + \sqrt{k+1})$$

So,

$$\frac{\sqrt{k+1}}{\sqrt{k+1}} + \frac{\sqrt{k+2}}{\sqrt{k+1}} > 2(\sqrt{k+2} - \sqrt{k+1})(\sqrt{k+2} + \sqrt{k+1})$$

Thus,

$$2 < 1 + \frac{\sqrt{k+2}}{\sqrt{k+1}}$$

This is true as $k \geq 1$. Therefore by PMI, our statement holds. ☽

3. Given

$$a_n = \begin{cases} 1 & n = 1 \\ 3 & n = 2 \\ a_{n-2} + 2a_{n-1} & n \geq 3 \end{cases}$$

Prove that a_n is odd for all integers $n \geq 1$.

Proof by Induction:
Base Cases:
Let $n = 1$ 1 is odd
Let $n = 2$ 3 is odd
So, our base cases hold.
Inductive Hypothesis: Assume $k \geq 2$ and that a_i is odd for all integers with $1 \leq i \leq k$.
Inductive Step: Consider, $n = k + 1$. So,

$$a_{k+1} = a_{k-1} + 2a_k$$

By our inductive hypothesis, a_{k-1} and a_k are odd. So $a_{k-1} = 2h + 1$ and $a_k = 2m + 1$ where $h, m \in \mathbb{Z}$. So,

$$a_{k+1} = 2h + 1 + 2(2m + 1)$$

$$= 2h + 1 + 4m + 2$$

$$= 2h + 4m + 2 + 1$$

$$= 2(h + 2m + 1) + 1$$

Therefore, a_{k+1} is odd. So, by principle of mathematical induction, our statement holds. ☽

4. Given

$$a_n = \begin{cases} 1 & n = 1 \\ 2 & n = 2 \\ \sum_{i=1}^{n-1} (i-1)a_i & n \geq 3 \end{cases}$$

Prove that $a_n = (n-1)!$ for all integers $n \geq 3$.

Proof by Induction:
Base Cases:

Let $n = 3$. Consider,

$$a_3 = \sum_{i=1}^{n-1} (i-1)a_i$$
$$= (1-1)(1) + (2-1)(2) = 0 + 2 = 2$$

Now consider, $(n-1)!$.

$$(n-1)! = (3-1)! = 2! = 2$$

Since $a_3 = 2$, $a_3 = (n-1)!$. So, our base case holds.
Inductive Hypothesis: Assume for some $k \geq 3$, $a_k = (k-1)!$
Inductive Step: Let $n = k+1$. So,

$$a_{k+1} = \sum_{i=1}^{k} (i-1)a_i$$

$$= \sum_{i=1}^{k-1} (i-1)a_i + (k-1)a_k$$

Note: $\sum_{i=1}^{k-1} (i-1)a_i = a_k$. So,

$$= a_k + (k-1)a_k$$

By our inductive hypothesis,

$$= (k-1)! + (k-1)(k-1)!$$
$$= (k-1)!(1 + k - 1)$$
$$= (k-1)!(k)$$
$$= k!$$

Therefore by principle of mathematical induction, our statement holds.
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5. Given

$$a_n = \begin{cases} 1 & n = 1 \\ 2 & n = 2 \\ \frac{a_{n-1}}{a_{n-2}} & n \geq 3 \end{cases}$$

(a) Prove that

$$a_n = \begin{cases} 1 & \text{if } n \equiv 1, 4 \pmod 6 \\ 2 & \text{if } n \equiv 2, 3 \pmod 6 \\ \frac{1}{2} & \text{if } n \equiv 0, 5 \pmod 6 \end{cases}$$

for all positive integers n.

Base Case:

Let $n = 1$. Since $a_n = 1$ and $n \equiv 1 \pmod 6$, this case holds.

Let $n = 2$. Since $a_n = 2$ and $n \equiv 2 \pmod 6$, this case holds.

Inductive Hypothesis: Assume for some $k \geq 2$ and $1 \leq i \leq k$,

$$a_i = \begin{cases} 1 & \text{if } i \equiv 1, 4 \pmod 6 \\ 2 & \text{if } i \equiv 2, 3 \pmod 6 \\ \frac{1}{2} & \text{if } i \equiv 0, 5 \pmod 6 \end{cases}$$

Inductive Step: Let $n = k + 1$. So, $a_{k+1} = \frac{a_k}{a_{k-1}}$. Consider the cases,

Case 1: Let $k - 1 \equiv 0 \pmod 6$ and $k \equiv 1 \pmod 6$. By our inductive hypothesis,

$$a_{k+1} = \frac{1}{\frac{1}{2}} = 2$$

Note if $k-1 \equiv 0 \pmod 6$ and $k \equiv 1 \pmod 6$, $k+1 \equiv 2 \pmod 6$. So this case holds.

Case 2: Let $k - 1 \equiv 1 \pmod 6$ and $k \equiv 2 \pmod 6$. By our inductive hypothesis,

$$a_{k+1} = \frac{2}{1} = 2$$

Note if $k-1 \equiv 1 \pmod 6$ and $k \equiv 2 \pmod 6$, $k+1 \equiv 3 \pmod 6$. So this case holds.

Case 3: Let $k - 1 \equiv 2 \pmod 6$ and $k \equiv 3 \pmod 6$. By our inductive hypothesis,

$$a_{k+1} = \frac{2}{2} = 1$$

Note if $k-1 \equiv 2 \pmod 6$ and $k \equiv 3 \pmod 6$, $k+1 \equiv 4 \pmod 6$. So this case holds.

Case 4: Let $k-1 \equiv 3 \pmod 6$ and $k \equiv 4 \pmod 6$. By our inductive hypothesis,

$$a_{k+1} = \frac{1}{2}$$

Note if $k-1 \equiv 3 \pmod 6$ and $k \equiv 4 \pmod 6$, $k+1 \equiv 5 \pmod 6$. So this case holds.

Case 5: Let $k-1 \equiv 4 \pmod 6$ and $k \equiv 5 \pmod 6$. By our inductive hypothesis,

$$a_{k+1} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

Note if $k-1 \equiv 4 \pmod 6$ and $k \equiv 5 \pmod 6$, $k+1 \equiv 0 \pmod 6$. So this case holds.

Case 5: Let $k-1 \equiv 5 \pmod 6$ and $k \equiv 0 \pmod 6$. By our inductive hypothesis,

$$a_{k+1} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

Note if $k-1 \equiv 5 \pmod 6$ and $k \equiv 0 \pmod 6$, $k+1 \equiv 1 \pmod 6$. So this case holds.

Therefore, by Principle of Mathematical Induction, our statement holds. ℵ

(b) Prove that for all nonnegative integers j, $\sum_{i=1}^{6} a_{j+i} = 7$

Base Case:

Let $j = 0$. So,

$$\sum_{i=1}^{6} a_i = a_1 + a_2 + a_3 + a_4 + a_5 + a_6$$

$$= 1 + 2 + 2 + 1 + \frac{1}{2} + \frac{1}{2} = 7.$$

Our base case holds.

Inductive Hypothesis: Assume for some $k \geq 0$, $\sum_{i=1}^{6} a_{k+i} = 7$

Inductive Step: Let $j = k + 1$. So,

$$\sum_{i=1}^{6} a_{k+1+i} = a_{k+2} + a_{k+3} + a_{k+4} + a_{k+5} + a_{k+6} + a_{k+7}$$

$$= (a_{k+1} + a_{k+2} + a_{k+3} + a_{k+4} + a_{k+5} + a_{k+6}) + a_{k+7} - a_{k+1}$$

$$= (\sum_{i=1}^{6} a_{k+i}) + a_{k+7} - a_{k+1}$$

By our inductive hypothesis,

$$= 7 + a_{k+7} - a_{k+1}.$$

Consider the cases,

Case 1: Let $k = 0$. $k + 1 \equiv 1 \pmod 6$ and $k + 7 \equiv 1 \pmod 6$. So,

$$= 7 + a_{k+7} - a_{k+1} = 7 + 1 - 1 = 7.$$

So this case holds.

Case 2: Let $k = 1$. $k + 1 \equiv 2 \pmod 6$ and $k + 7 \equiv 2 \pmod 6$. So,

$$= 7 + a_{k+7} - a_{k+1} = 7 + 2 - 2 = 7.$$

So this case holds.

Case 3: Let $k = 2$. $k + 1 \equiv 3 \pmod 6$ and $k + 7 \equiv 3 \pmod 6$. So,

$$= 7 + a_{k+7} - a_{k+1} = 7 + 2 - 2 = 7.$$

So this case holds.

Case 4: Let $k = 3$. $k + 1 \equiv 4 \pmod 6$ and $k + 7 \equiv 4 \pmod 6$. So,

$$= 7 + a_{k+7} - a_{k+1} = 7 + 1 - 1 = 7.$$

So this case holds.

Case 5: Let $k = 4$. $k+1 \equiv 5 \pmod 6$ and $k+7 \equiv 5 \pmod 6$. So,

$$= 7 + a_{k+7} - a_{k+1} = 7 + \frac{1}{2} - \frac{1}{2} = 7.$$

So this case holds.

Case 6: Let $k = 5$. $k+1 \equiv 0 \pmod 6$ and $k+7 \equiv 0 \pmod 6$. So,

$$= 7 + a_{k+7} - a_{k+1} = 7 + \frac{1}{2} - \frac{1}{2} = 7.$$

So this case holds.

Therefore, by Principle of Mathematical Induction, our statement holds. ☽

6. Use Constructive Induction to find constants A, B, C for

$$\sum_{i=1}^{n} 4i - 3 = An^2 + Bn + C.$$

Let us guess that

$$\sum_{i=1}^{n} 4i - 3 = An^2 + Bn + C.$$

Base Case: Let $n = 1$. So,

$$\sum_{i=1}^{1} 4i - 3 = 1$$

$$1 = A(1)^2 + B(1) + C$$

$$1 = A + B + C$$

Inductive Hypothesis: Assume for some $n \geq 1$,

$$\sum_{i=1}^{n} 4i - 3 = An^2 + Bn + C.$$

Inductive Step: Consider $n+1$. So,

$$\sum_{i=1}^{n+1} 4i-3 = \sum_{i=1}^{n} 4i-3+4(n+1)-3$$

By our Inductive Hypothesis,

$$An^2+Bn+C+4(n+1)-3.$$

So,

$$\begin{aligned} An^2+Bn+C+4(n+1)-3 &= A(n+1)^2+B(n+1)+C \\ An^2+Bn+C+4n+1 &= A(n^2+2n+1)+B(n+1)+C \\ An^2+Bn+C+4n+1 &= An^2+2An+A+Bn+B+C \\ 4n+1 &= 2An+A+B \end{aligned}$$

Thus,

$$\begin{aligned} 4 &= 2(A) \\ 1 &= A+B \end{aligned}$$

So, $A=2$ and $B=-1$. From the Base Case, we had

$$1 = A+B+C$$

So, $C=0$. Thus our solution give us

$$\sum_{i=1}^{n} 4i-3 = 2n^2+-n.$$

7. Use Constructive Induction to find constants A, B, C, D for

$$\sum_{i=1}^{n} i(i+2) = An^3+Bn^2+Cn+D.$$

Let is guess that

$$a_n = An^3+Bn^2+Cn+D.$$

Base Case:
Let $n = 1$. So,

$$\sum_{i=1}^{1} i(i+2) = 1(1+2) = 3$$

$$A + B + C + D = 3$$

Inductive Hypothesis:
Assume for some $n \geq 1$,

$$\sum_{i=1}^{n} i(i+2) = An^3 + Bn^2 + Cn + D$$

Inductive Step:
Consider $n + 1$.

$$\sum_{i=1}^{n+1} i(i+2) = \sum_{i=1}^{n} i(i+2) + (n+1)(n+3)$$

By inductive hypothesis,

$$An^3+Bn^2+Cn+D+(n+1)(n+3) = A(n+1)^3+B(n+1)^2+C(n+1)+D$$

$$An^3+Bn^2+Cn+D+n^2+4n+3 = A(n^3+3n^2+3n+1)+B(n^2+2n+1)+C(n+1)+D$$

$$An^3+Bn^2+Cn+D+n^2+4n+3 = An^3+3An^2+3An+A+Bn^2+2Bn+B+Cn+C+D$$

$$n^2 + 4n + 3 = 3An^2 + 3An + A + 2Bn + B + C$$

Therefore we have the equations,

$$1 = 3A$$

$$4 = 3A + 2B$$

$$3 = A + B + C$$

Thus, $A = \frac{1}{3}$, $B = \frac{3}{2}$, $C = \frac{7}{6}$, and $D = 0$. Hence,

$$\sum_{i=1}^{n} i(i+2) = \frac{1}{3}n^3 + \frac{3}{2}n^2 + \frac{7}{6}n.$$

8. Use Constructive Induction to find constants A, B, C for

$$a_n = \begin{cases} 1 & n = 1 \\ 4 & n = 2 \\ 9 & n = 3 \\ a_{n-1} - a_{n-2} + a_{n-3} + 2(2n-3) & n \geq 4 \end{cases}$$

such that $a_n = An^2 + Bn + C$.

Let us guess that

$$a_n = An^2 + Bn + C$$

Base Case:

Consider $n = 1$,

$$1 = A(1)^2 + B(1) + C$$
$$1 = A + B + C$$

Consider $n = 2$,

$$4 = A(2)^2 + B(2) + C$$
$$4 = 4A + 2B + C$$

Consider $n = 3$,

$$1 = A(3)^2 + B(3) + C$$
$$9 = 9A + 3B + C$$

Inductive Hypothesis: Assume for some $n \geq 3$ and $1 \leq i \leq n$,

$$a_i = Ai^2 + Bi + C$$

Inductive Step: Consider $n + 1$. So,

$$a_{n+1} = a_n - a_{n-1} + a_{n-2} + 2(2(n+1) - 3)$$

By our inductive hypothesis,

$$a_n - a_{n-1} + a_{n-2} + 2(2n - 3)$$

$$= (An^2+Bn+C)-(A(n-1)^2+B(n-1)+C)+(A(n-2)^2+B(n-2)+C)+4n-2$$
$$= (An^2+Bn+C)+(-An^2+2An-A-Bn+B-C)+(An^2-4An+4A+Bn-2B+C)+4n-2$$
$$= An^2 - 2An + 3A + Bn - B + C + 4n - 2$$

Therefore,

$$An^2 - 2An + 3A + Bn - B + C + 4n - 2 = A(n+1)^2 + B(n+1) + C$$

$$An^2+(-2An+Bn+4n)+(3A-B+C-2) = A(n^2+2n+1)+B(n+1)+C$$
$$An^2+(-2An+Bn+4n)+(3A-B+C-2) = An^2+2An+A+Bn+B+C$$
$$An^2+(-2An+Bn+4n)+(3A-B+C-2) = An^2+(2An+Bn)+(A+B+C)$$
$$(-2An+4n)+(3A-B-2) = (2An)+(A+B)$$
$$(4n)+(-2) = (4An)+(-2A+2B).$$

So, we have the equations

$$4 = 4A$$

and

$$-2 = -2A + 2B.$$

Thus, $A = 1$ and $B = 0$. Now we must go back to our base cases. So,

$$1 = 1 + 0 + C$$

$$4 = 4(1) + 2(0) + C$$
$$9 = 9(1) + 3(0) + C.$$

Therefore, $C = 0$. Hence, $a_n = n^2$.

9. Use Constructive Induction to a constant A bound for

$$\sum_{i=1}^{n} \frac{1}{(i+2)(i+3)}$$

such that $a_n \leq An$

Let us guess that

$$a_n \leq An$$

Base Case:
Consider $n = 1$,

$$\sum_{i=1}^{1} \frac{1}{(i+2)(i+3)} = \frac{1}{(1+2)(1+3)}$$

$$= \frac{1}{(3)(4)}$$

So,

$$\frac{1}{12} \leq An$$

Inductive Hypothesis:
Assume for some $n \geq 1$,

$$\sum_{i=1}^{n} \frac{1}{(i+2)(i+3)} \leq An$$

Inductive Step:
Consider $n+1$. So,

$$\sum_{i=1}^{n+1} \frac{1}{(i+2)(i+3)} = \sum_{i=1}^{n} \frac{1}{(i+2)(i+3)} + \frac{1}{[(n+1)+2][(n+1)+3]}$$

$$= \sum_{i=1}^{n} \frac{1}{(i+2)(i+3)} + \frac{1}{(n+3)(n+4)}$$

By inductive hypothesis,

$$\sum_{i=1}^{n} \frac{1}{(i+2)(i+3)} + \frac{1}{(n+3)(n+4)} \leq An + \frac{1}{(n+3)(n+4)}.$$

Therefore,

$$An + \frac{1}{(n+3)(n+4)} \leq A(n+1)$$

$$An + \frac{1}{(n+3)(n+4)} \leq An + A$$

$$\frac{1}{(n+3)(n+4)} \leq A.$$

Therefore, $A = \frac{1}{12}$. Hence,

$$\sum_{i=1}^{n} \frac{1}{(i+2)(i+3)} \leq \frac{1}{12} n$$