The BEE Sequence

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Erik Metz did the hard math.

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4. Emily said I cannot use that so long as she is my TA.
5. I'll use the bee story when I teach 250H in Spring 2026.

The BEE Sequence
Mod 2

The First Few Values Mod 2

All $\equiv$ in this section are mod 2.

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 2$

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Vote YES, NO, UNKNOWN TO G-K-M?

Lets Try To Spot a Pattern!

First some empirical observations.

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What do you notice about $\boldsymbol{n}$, $\boldsymbol{a_n}$ and **Mod 2**? Discuss.

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What do you notice about ***n***, $\boldsymbol{a_n}$ and **Mod 2**? Discuss.

If $n \equiv 1$ then $a_n \equiv 1$. Lets Prove This!

$n \equiv 1 \to a_n \equiv 1$. Induction

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IB $a_1 = 1 \equiv 1$.

$n \equiv 1 \rightarrow a_n \equiv 1$. Induction

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$a_{2n-1} \equiv 1$ by the IH.

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$a_{2n-1} \equiv 1$ by the IH.

Hence $a_{2n+1} \equiv a_{2n-1} \equiv 1$.

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Since for all ODD m, $a_m \equiv 1$ we have
$a_{2n} = a_{2n-1} + a_n \equiv 1 + a_n$.

$(\exists^\infty n)[a_n \equiv 0]$

This one does not need induction.

Since for all ODD m, $a_m \equiv 1$ we have
$a_{2n} = a_{2n-1} + a_n \equiv 1 + a_n$.

If n is odd then we have
$a_{2n} = a_{2n-1} + a_n \equiv 1 + a_n \equiv 1 + 1 \equiv 0$.

$(\exists^\infty n)[a_n \equiv 0]$

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Since for all ODD m, $a_m \equiv 1$ we have
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$a_{2n} = a_{2n-1} + a_n \equiv 1 + a_n \equiv 1 + 1 \equiv 0$.

Upshot For all k

$$a_{2(2k+1)} = a_{2(2k+1)-1} + a_{2k+1} \equiv 1 + 1 \equiv 0.$$

A Proof that Does Not Need Induction

Thm For all n
$a_{n+1} \equiv 0 \vee a_{2n+1} \equiv 0 \vee a_{2n+2} \equiv 0$.

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Pf

Case 1 $a_{n+1} \equiv 0$. DONE.

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Thm For all n

$a_{n+1} \equiv 0 \vee a_{2n+1} \equiv 0 \vee a_{2n+2} \equiv 0.$

Pf

Case 1 $a_{n+1} \equiv 0$. DONE.

Case 2 $a_{2n+1} \equiv 0$. DONE.

A Proof that Does Not Need Induction

Thm For all n
$a_{n+1} \equiv 0 \vee a_{2n+1} \equiv 0 \vee a_{2n+2} \equiv 0$.
Pf

Case 1 $a_{n+1} \equiv 0$. DONE.

Case 2 $a_{2n+1} \equiv 0$. DONE.

Case 3 $a_{n+1} \equiv 1$ and $a_{2n+1} \equiv 1$.

A Proof that Does Not Need Induction

Thm For all n
$a_{n+1} \equiv 0 \vee a_{2n+1} \equiv 0 \vee a_{2n+2} \equiv 0$.
Pf

Case 1 $a_{n+1} \equiv 0$. DONE.

Case 2 $a_{2n+1} \equiv 0$. DONE.

Case 3 $a_{n+1} \equiv 1$ and $a_{2n+1} \equiv 1$.

$$a_{2n+2} = a_{2n+1} + a_{n+1} \equiv 1 + 1 \equiv 0.$$

Which Proof Did You Like Better?

Vote Which proof did you like better?

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1. The proof where we first show $a_m \equiv 1$ for odd m, and then show $a_{2(2k+1)} \equiv 0$.

Which Proof Did You Like Better?

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1. The proof where we first show $a_m \equiv 1$ for odd m, and then show $a_{2(2k+1)} \equiv 0$.
2. The proof where we showed that, for all n, $a_{n+1} \equiv 0 \vee a_{2n+1} \equiv 0 \vee a_{2n+2} \equiv 0$.

The BEE Sequence Mod 3

The First Few Values Mod 3

In this section all $\equiv$ are mod 3.

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What do you notice about n, a_n and Mod 3? Discuss. NOTHING!

$\exists^\infty n$ with $a_n \equiv 0$

Thm For all n
$a_{n+1} \equiv 0 \vee a_{2n+1} \equiv 0 \vee a_{2n+2} \equiv 0 \vee a_{2n+3} \equiv 0.$

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Case 3a $a_{2n+1} \equiv 1$

$\exists^{\infty} n$ with $a_n \equiv 0$

Thm For all n

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Pf

Case 1 $a_{n+1} \equiv 0$. DONE.

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$\exists^\infty n$ with $a_n \equiv 0$

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The BEE Sequence Mod 4

The First Few Values Mod 4

In this section all $\equiv$ are mod 4.

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3. **Bill** wrote it up.

The BEE Sequence Mod 5

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5	$a_5 = a_4 + a_2$	7	2
6	$a_6 = a_5 + a_3$	10	0
7	$a_7 = a_6 + a_3$	13	3
8	$a_8 = a_7 + a_4$	18	3

The First Few Values Mod 5

In this section all $\equiv$ are mod 5.

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 5$
1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2
3	$a_3 = a_2 + a_1$	3	3
4	$a_4 = a_3 + a_2$	5	0
5	$a_5 = a_4 + a_2$	7	2
6	$a_6 = a_5 + a_3$	10	0
7	$a_7 = a_6 + a_3$	13	3
8	$a_8 = a_7 + a_4$	18	3
9	$a_9 = a_8 + a_4$	23	3

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4	$a_4 = a_3 + a_2$	5	0
5	$a_5 = a_4 + a_2$	7	2
6	$a_6 = a_5 + a_3$	10	0
7	$a_7 = a_6 + a_3$	13	3
8	$a_8 = a_7 + a_4$	18	3
9	$a_9 = a_8 + a_4$	23	3
10	$a_{10} = a_9 + a_5$	30	0

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1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2
3	$a_3 = a_2 + a_1$	3	3
4	$a_4 = a_3 + a_2$	5	0
5	$a_5 = a_4 + a_2$	7	2
6	$a_6 = a_5 + a_3$	10	0
7	$a_7 = a_6 + a_3$	13	3
8	$a_8 = a_7 + a_4$	18	3
9	$a_9 = a_8 + a_4$	23	3
10	$a_{10} = a_9 + a_5$	30	0
11	$a_{11} = a_{10} + a_5$	37	2

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3	$a_3 = a_2 + a_1$	3	3
4	$a_4 = a_3 + a_2$	5	0
5	$a_5 = a_4 + a_2$	7	2
6	$a_6 = a_5 + a_3$	10	0
7	$a_7 = a_6 + a_3$	13	3
8	$a_8 = a_7 + a_4$	18	3
9	$a_9 = a_8 + a_4$	23	3
10	$a_{10} = a_9 + a_5$	30	0
11	$a_{11} = a_{10} + a_5$	37	2

No pattern here. But $a_4 \equiv a_6 \equiv 0$.

Lets Use $a_6 \equiv 0$

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$a_{12} = a_{11} + a_6 \equiv a_{11}$

Lets Use $a_6 \equiv 0$

We will use $a_6 \equiv 0$ to get some larger n with $a_n \equiv 0$.

a_6 is used for both a_{12} and a_{13}.

$a_{12} = a_{11} + a_6 \equiv a_{11}$

$a_{13} = a_{12} + a_6 \equiv a_{12}$.

Lets Use $a_6 \equiv 0$

We will use $a_6 \equiv 0$ to get some larger n with $a_n \equiv 0$.

a_6 is used for both a_{12} and a_{13}.

$a_{12} = a_{11} + a_6 \equiv a_{11}$
$a_{13} = a_{12} + a_6 \equiv a_{12}$.
So we get

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$a_{13} = a_{12} + a_6 \equiv a_{12}$.
So we get

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$a_{13} = a_{12} + a_6 \equiv a_{12}$.
So we get

$$a_{11} \equiv a_{12} \equiv a_{13}.$$

Lets use that!

Lets Use $a_{11} \equiv a_{12} \equiv a_{13}$

Let $a_{11} \equiv a_{12} \equiv a_{13} \equiv r$

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The sequence uses a_{11} for a_{22} and a_{23}

Lets Use $a_{11} \equiv a_{12} \equiv a_{13}$

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The sequence uses a_{11} for a_{22} and a_{23}

The sequence uses a_{12} for a_{24} and a_{25}

Lets Use $a_{11} \equiv a_{12} \equiv a_{13}$

Let $a_{11} \equiv a_{12} \equiv a_{13} \equiv r$

The sequence uses a_{11} for a_{22} and a_{23}
The sequence uses a_{12} for a_{24} and a_{25}
The sequence uses a_{13} for a_{26} and a_{27}

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The sequence uses a_{11} for a_{22} and a_{23}
The sequence uses a_{12} for a_{24} and a_{25}
The sequence uses a_{13} for a_{26} and a_{27}

$a_{22} = a_{21} + a_{11} \equiv a_{21} + r$

Lets Use $a_{11} \equiv a_{12} \equiv a_{13}$

Let $a_{11} \equiv a_{12} \equiv a_{13} \equiv r$

The sequence uses a_{11} for a_{22} and a_{23}
The sequence uses a_{12} for a_{24} and a_{25}
The sequence uses a_{13} for a_{26} and a_{27}

$a_{22} = a_{21} + a_{11} \equiv a_{21} + r$
$a_{23} = a_{22} + a_{11} \equiv a_{21} + r + a_{11} \equiv a_{21} + 2r$

Lets Use $a_{11} \equiv a_{12} \equiv a_{13}$

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The sequence uses a_{11} for a_{22} and a_{23}
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$$a_{24} = a_{23} + a_{12} \equiv a_{21} + 2r + a_{12} \equiv a_{21} + 3r$$

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Continued Next Slide.

Lets Use $a_{11} \equiv a_{12} \equiv a_{13}$ (cont)

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Lets Use $a_{11} \equiv a_{12} \equiv a_{13}$ (cont)

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Case 0 $r \equiv 0$. DONE, $a_{11} \equiv 0$. Later cases assume $r \not\equiv 0$.

Lets Use $a_{11} \equiv a_{12} \equiv a_{13}$ (cont)

Let $a_{11} \equiv a_{12} \equiv a_{13} \equiv r$

$a_{22} = a_{21} + a_{11} \equiv a_{21} + r$
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Case 1 $a_{21} \equiv 0$. DONE. Later cases assume $a_{21} \not\equiv 0$.

Lets Use $a_{11} \equiv a_{12} \equiv a_{13}$ (cont)

Let $a_{11} \equiv a_{12} \equiv a_{13} \equiv r$

$a_{22} = a_{21} + a_{11} \equiv a_{21} + r$
$a_{23} = a_{22} + a_{11} \equiv a_{21} + 2r$
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Case 1 $a_{21} \equiv 0$. DONE. Later cases assume $a_{21} \not\equiv 0$.

Case 2 Whats left.
One of $a_{21} + r$, $a_{21} + 2r$, $a_{21} + 3r$, $a_{21} + 4r$ is $\equiv 0$.

Can We Generalize This Approach? Yes

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We prove $(\exists^\infty n)[a_n \equiv 0]$.

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IB For $n = 1$ take $i_1 = 6$. Note that $a_6 \equiv 0$.

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IB For $n = 1$ take $i_1 = 6$. Note that $a_6 \equiv 0$.
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IS Let $i_n = m$. We use $a_m \equiv 0$ to show $(\exists m' > m)[a_{m'} \equiv 0]$.

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Can We Generalize This Approach? Yes (Cont)

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We will use $a_m \equiv 0$ to get some larger m' with $a_{m'} \equiv 0$.

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$a_{2m} = a_{2m-1} + a_m \equiv a_{2m-1}$

Can We Generalize This Approach? Yes (Cont)

We will use $a_m \equiv 0$ to get some larger m' with $a_{m'} \equiv 0$.

a_m is used for both a_{2m} and a_{2m+1}.

$a_{2m} = a_{2m-1} + a_m \equiv a_{2m-1}$
$a_{2m+1} = a_{2m} + a_m \equiv a_{2m-1}$.

Can We Generalize This Approach? Yes (Cont)

We will use $a_m \equiv 0$ to get some larger m' with $a_{m'} \equiv 0$.

a_m is used for both a_{2m} and a_{2m+1}.

$a_{2m} = a_{2m-1} + a_m \equiv a_{2m-1}$
$a_{2m+1} = a_{2m} + a_m \equiv a_{2m-1}$.
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Can We Generalize This Approach? Yes (Cont)

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a_m is used for both a_{2m} and a_{2m+1}.

$a_{2m} = a_{2m-1} + a_m \equiv a_{2m-1}$
$a_{2m+1} = a_{2m} + a_m \equiv a_{2m-1}$.
So we get

$$a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}.$$

Can We Generalize This Approach? Yes (Cont)

We will use $a_m \equiv 0$ to get some larger m' with $a_{m'} \equiv 0$.

a_m is used for both a_{2m} and a_{2m+1}.

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So we get

$$a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}.$$

Lets use that!

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

Let $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1} \equiv r$

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

Let $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1} \equiv r$

The sequence uses a_{2m-1} for a_{4m-2} and a_{4m-1}

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

Let $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1} \equiv r$

The sequence uses a_{2m-1} for a_{4m-2} and a_{4m-1}

The sequence uses a_{2m} for a_{4m} and a_{4m+1}

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

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The sequence uses a_{2m-1} for a_{4m-2} and a_{4m-1}

The sequence uses a_{2m} for a_{4m} and a_{4m+1}

The sequence uses a_{2m+1} for a_{4m+2} and a_{4m+3}

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

Let $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1} \equiv r$

The sequence uses a_{2m-1} for a_{4m-2} and a_{4m-1}
The sequence uses a_{2m} for a_{4m} and a_{4m+1}
The sequence uses a_{2m+1} for a_{4m+2} and a_{4m+3}

$a_{4m-2} = a_{4m-3} + a_{2m-1} \equiv a_{4m-3} + r$

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

Let $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1} \equiv r$

The sequence uses a_{2m-1} for a_{4m-2} and a_{4m-1}
The sequence uses a_{2m} for a_{4m} and a_{4m+1}
The sequence uses a_{2m+1} for a_{4m+2} and a_{4m+3}

$a_{4m-2} = a_{4m-3} + a_{2m-1} \equiv a_{4m-3} + r$
$a_{4m-1} = a_{4m-2} + a_{2m-1} \equiv a_{4m-3} + 2r$

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

Let $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1} \equiv r$

The sequence uses a_{2m-1} for a_{4m-2} and a_{4m-1}
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The sequence uses a_{2m+1} for a_{4m+2} and a_{4m+3}

$a_{4m-2} = a_{4m-3} + a_{2m-1} \equiv a_{4m-3} + r$
$a_{4m-1} = a_{4m-2} + a_{2m-1} \equiv a_{4m-3} + 2r$
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Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

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The sequence uses a_{2m-1} for a_{4m-2} and a_{4m-1}
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$$a_{4m+1} = a_{4m} + a_{2m} \equiv a_{4m-3} + 4r$$

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

Let $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1} \equiv r$

The sequence uses a_{2m-1} for a_{4m-2} and a_{4m-1}
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The sequence uses a_{2m+1} for a_{4m+2} and a_{4m+3}

$a_{4m-2} = a_{4m-3} + a_{2m-1} \equiv a_{4m-3} + r$
$a_{4m-1} = a_{4m-2} + a_{2m-1} \equiv a_{4m-3} + 2r$
$a_{4m} = a_{4m-1} + a_{2m} \equiv a_{4m-3} + 3r$
$a_{4m+1} = a_{4m} + a_{2m} \equiv a_{4m-3} + 4r$

Continued Next Slide.

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

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$$a_{4m-2} = a_{4m-3} + a_{2m-1} \equiv a_{4m-3} + r$$

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

$$a_{4m-2} = a_{4m-3} + a_{2m-1} \equiv a_{4m-3} + r$$
$$a_{4m-1} = a_{4m-2} + a_{2m-1} \equiv a_{4m-3} + 2r$$

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

$$a_{4m-2} = a_{4m-3} + a_{2m-1} \equiv a_{4m-3} + r$$
$$a_{4m-1} = a_{4m-2} + a_{2m-1} \equiv a_{4m-3} + 2r$$
$$a_{4m} = a_{4m-1} + a_{2m} \equiv a_{4m-3} + 3r$$

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$$a_{4m-2} = a_{4m-3} + a_{2m-1} \equiv a_{4m-3} + r$$
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$$a_{4m} = a_{4m-1} + a_{2m} \equiv a_{4m-3} + 3r$$
$$a_{4m+1} = a_{4m} + a_{2m} \equiv a_{4m-3} + 4r$$

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

$a_{4m-2} = a_{4m-3} + a_{2m-1} \equiv a_{4m-3} + r$
$a_{4m-1} = a_{4m-2} + a_{2m-1} \equiv a_{4m-3} + 2r$
$a_{4m} = a_{4m-1} + a_{2m} \equiv a_{4m-3} + 3r$
$a_{4m+1} = a_{4m} + a_{2m} \equiv a_{4m-3} + 4r$
Case 0 $r \equiv 0$. DONE, $a_{2m-1} \equiv 0$. Later cases assume $r \not\equiv 0$.

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

$a_{4m-2} = a_{4m-3} + a_{2m-1} \equiv a_{4m-3} + r$
$a_{4m-1} = a_{4m-2} + a_{2m-1} \equiv a_{4m-3} + 2r$
$a_{4m} = a_{4m-1} + a_{2m} \equiv a_{4m-3} + 3r$
$a_{4m+1} = a_{4m} + a_{2m} \equiv a_{4m-3} + 4r$

Case 0 $r \equiv 0$. DONE, $a_{2m-1} \equiv 0$. Later cases assume $r \not\equiv 0$.

Case 1 $a_{4m-3} \equiv 0$. DONE. Later cases assume $a_{4m-3} \not\equiv 0$.

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

$a_{4m-2} = a_{4m-3} + a_{2m-1} \equiv a_{4m-3} + r$
$a_{4m-1} = a_{4m-2} + a_{2m-1} \equiv a_{4m-3} + 2r$
$a_{4m} = a_{4m-1} + a_{2m} \equiv a_{4m-3} + 3r$
$a_{4m+1} = a_{4m} + a_{2m} \equiv a_{4m-3} + 4r$

Case 0 $r \equiv 0$. DONE, $a_{2m-1} \equiv 0$. Later cases assume $r \not\equiv 0$.

Case 1 $a_{4m-3} \equiv 0$. DONE. Later cases assume $a_{4m-3} \not\equiv 0$.

Case 2 Whats left.
One of $a_{4m-3} + r$, $a_{4m-3} + 2r$, $a_{4m-3} + 3r$, $a_{4m-3} + 4r$ is $\equiv 0$.

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

$a_{4m-2} = a_{4m-3} + a_{2m-1} \equiv a_{4m-3} + r$
$a_{4m-1} = a_{4m-2} + a_{2m-1} \equiv a_{4m-3} + 2r$
$a_{4m} = a_{4m-1} + a_{2m} \equiv a_{4m-3} + 3r$
$a_{4m+1} = a_{4m} + a_{2m} \equiv a_{4m-3} + 4r$

Case 0 $r \equiv 0$. DONE, $a_{2m-1} \equiv 0$. Later cases assume $r \not\equiv 0$.

Case 1 $a_{4m-3} \equiv 0$. DONE. Later cases assume $a_{4m-3} \not\equiv 0$.

Case 2 Whats left.
One of $a_{4m-3} + r$, $a_{4m-3} + 2r$, $a_{4m-3} + 3r$, $a_{4m-3} + 4r$ is $\equiv 0$.
So we have an $m' > m$ such that $a_{m'} \equiv 0$.

The BEE Sequence Mod 7

The First Few Values Mod 7

The First Few Values Mod 7

In this section all $\equiv$ are mod 7

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 7$

The First Few Values Mod 7

In this section all $\equiv$ are mod 7

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 7$
1	a_1	1	1

The First Few Values Mod 7

In this section all $\equiv$ are mod 7

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 7$
1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2

The First Few Values Mod 7

In this section all $\equiv$ are mod 7

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 7$
1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2
3	$a_3 = a_2 + a_1$	3	3

The First Few Values Mod 7

In this section all $\equiv$ are mod 7

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 7$
1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2
3	$a_3 = a_2 + a_1$	3	3
4	$a_4 = a_3 + a_2$	5	5

The First Few Values Mod 7

In this section all $\equiv$ are mod 7

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 7$
1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2
3	$a_3 = a_2 + a_1$	3	3
4	$a_4 = a_3 + a_2$	5	5
5	$a_5 = a_4 + a_2$	7	0

The First Few Values Mod 7

In this section all $\equiv$ are mod 7

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 7$
1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2
3	$a_3 = a_2 + a_1$	3	3
4	$a_4 = a_3 + a_2$	5	5
5	$a_5 = a_4 + a_2$	7	0
6	$a_6 = a_5 + a_3$	10	3

The First Few Values Mod 7

In this section all $\equiv$ are mod 7

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 7$
1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2
3	$a_3 = a_2 + a_1$	3	3
4	$a_4 = a_3 + a_2$	5	5
5	$a_5 = a_4 + a_2$	7	0
6	$a_6 = a_5 + a_3$	10	3
7	$a_7 = a_6 + a_3$	13	6

The First Few Values Mod 7

In this section all $\equiv$ are mod 7

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 7$
1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2
3	$a_3 = a_2 + a_1$	3	3
4	$a_4 = a_3 + a_2$	5	5
5	$a_5 = a_4 + a_2$	7	0
6	$a_6 = a_5 + a_3$	10	3
7	$a_7 = a_6 + a_3$	13	6
8	$a_8 = a_7 + a_4$	18	4

The First Few Values Mod 7

In this section all $\equiv$ are mod 7

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 7$
1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2
3	$a_3 = a_2 + a_1$	3	3
4	$a_4 = a_3 + a_2$	5	5
5	$a_5 = a_4 + a_2$	7	0
6	$a_6 = a_5 + a_3$	10	3
7	$a_7 = a_6 + a_3$	13	6
8	$a_8 = a_7 + a_4$	18	4
9	$a_9 = a_8 + a_4$	23	2

The First Few Values Mod 7

In this section all $\equiv$ are mod 7

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 7$
1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2
3	$a_3 = a_2 + a_1$	3	3
4	$a_4 = a_3 + a_2$	5	5
5	$a_5 = a_4 + a_2$	7	0
6	$a_6 = a_5 + a_3$	10	3
7	$a_7 = a_6 + a_3$	13	6
8	$a_8 = a_7 + a_4$	18	4
9	$a_9 = a_8 + a_4$	23	2
10	$a_{10} = a_9 + a_5$	30	2

The First Few Values Mod 7

In this section all $\equiv$ are mod 7

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 7$
1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2
3	$a_3 = a_2 + a_1$	3	3
4	$a_4 = a_3 + a_2$	5	5
5	$a_5 = a_4 + a_2$	7	0
6	$a_6 = a_5 + a_3$	10	3
7	$a_7 = a_6 + a_3$	13	6
8	$a_8 = a_7 + a_4$	18	4
9	$a_9 = a_8 + a_4$	23	2
10	$a_{10} = a_9 + a_5$	30	2
11	$a_{11} = a_{10} + a_5$	37	2

No pattern here. But $a_5 \equiv 0 \pmod 7$.

Lets Try Same Approach as Mod 5

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We will use $a_m \equiv 0$ to get some larger m' with $a_{m'} \equiv 0$.

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$a_{2m} = a_{2m-1} + a_m \equiv a_{2m-1}$

Lets Try Same Approach as Mod 5

We will use $a_m \equiv 0$ to get some larger m' with $a_{m'} \equiv 0$.

a_m is used for both a_{2m} and a_{2m+1}.

$a_{2m} = a_{2m-1} + a_m \equiv a_{2m-1}$

$a_{2m+1} = a_{2m} + a_m \equiv a_{2m-1}$

Lets Try Same Approach as Mod 5

We will use $a_m \equiv 0$ to get some larger m' with $a_{m'} \equiv 0$.

a_m is used for both a_{2m} and a_{2m+1}.

$a_{2m} = a_{2m-1} + a_m \equiv a_{2m-1}$
$a_{2m+1} = a_{2m} + a_m \equiv a_{2m-1}$
So we get

$$a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}.$$

Lets Try Same Approach as Mod 5

We will use $a_m \equiv 0$ to get some larger m' with $a_{m'} \equiv 0$.

a_m is used for both a_{2m} and a_{2m+1}.

$a_{2m} = a_{2m-1} + a_m \equiv a_{2m-1}$
$a_{2m+1} = a_{2m} + a_m \equiv a_{2m-1}$
So we get

$$a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}.$$

Lets use that!

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

WORK ON THIS IN GROUPS.

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

Let $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1} \equiv r$

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

Let $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1} \equiv r$

$a_{4m-2} = a_{4m-3} + a_{2m-1} \equiv a_{4m-3} + r$

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

Let $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1} \equiv r$

$$a_{4m-2} = a_{4m-3} + a_{2m-1} \equiv a_{4m-3} + r$$
$$a_{4m-1} = a_{4m-2} + a_{2m-1} \equiv a_{4m-3} + 2r$$

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

Let $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1} \equiv r$

$a_{4m-2} = a_{4m-3} + a_{2m-1} \equiv a_{4m-3} + r$
$a_{4m-1} = a_{4m-2} + a_{2m-1} \equiv a_{4m-3} + 2r$
$a_{4m} = a_{4m-1} + a_{2m} \equiv a_{4m-3} + 3r$

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

Let $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1} \equiv r$

$$a_{4m-2} = a_{4m-3} + a_{2m-1} \equiv a_{4m-3} + r$$
$$a_{4m-1} = a_{4m-2} + a_{2m-1} \equiv a_{4m-3} + 2r$$
$$a_{4m} = a_{4m-1} + a_{2m} \equiv a_{4m-3} + 3r$$
$$a_{4m+1} = a_{4m} + a_{2m} \equiv a_{4m-3} + 4r$$

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

Let $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1} \equiv r$

$$a_{4m-2} = a_{4m-3} + a_{2m-1} \equiv a_{4m-3} + r$$
$$a_{4m-1} = a_{4m-2} + a_{2m-1} \equiv a_{4m-3} + 2r$$
$$a_{4m} = a_{4m-1} + a_{2m} \equiv a_{4m-3} + 3r$$
$$a_{4m+1} = a_{4m} + a_{2m} \equiv a_{4m-3} + 4r$$
$$a_{4m+2} = a_{4m+1} + a_{2m+1} \equiv a_{4m-3} + 5r$$

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

Let $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1} \equiv r$

$$a_{4m-2} = a_{4m-3} + a_{2m-1} \equiv a_{4m-3} + r$$
$$a_{4m-1} = a_{4m-2} + a_{2m-1} \equiv a_{4m-3} + 2r$$
$$a_{4m} = a_{4m-1} + a_{2m} \equiv a_{4m-3} + 3r$$
$$a_{4m+1} = a_{4m} + a_{2m} \equiv a_{4m-3} + 4r$$
$$a_{4m+2} = a_{4m+1} + a_{2m+1} \equiv a_{4m-3} + 5r$$
$$a_{4m+3} = a_{4m+2} + a_{2m+1} \equiv a_{4m-3} + 6r$$

Lets Use $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}$

Let $a_{2m-1} \equiv a_{2m} \equiv a_{2m+1} \equiv r$

$a_{4m-2} = a_{4m-3} + a_{2m-1} \equiv a_{4m-3} + r$
$a_{4m-1} = a_{4m-2} + a_{2m-1} \equiv a_{4m-3} + 2r$
$a_{4m} = a_{4m-1} + a_{2m} \equiv a_{4m-3} + 3r$
$a_{4m+1} = a_{4m} + a_{2m} \equiv a_{4m-3} + 4r$
$a_{4m+2} = a_{4m+1} + a_{2m+1} \equiv a_{4m-3} + 5r$
$a_{4m+3} = a_{4m+2} + a_{2m+1} \equiv a_{4m-3} + 6r$
Since have $\{r, 2r, 3r, 4r, 5r, 6r\}$ proof is similar to Mod 5.

The BEE Sequence Mod 9

The First Few Values Mod 9

(We skip mod 8 since mod 4 didn't work).
In this section all $\equiv$ are mod 9.

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 9$

The First Few Values Mod 9

(We skip mod 8 since mod 4 didn't work).
In this section all $\equiv$ are mod 9.

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 9$
1	a_1	1	1

The First Few Values Mod 9

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In this section all $\equiv$ are mod 9.

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 9$
1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2

The First Few Values Mod 9

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n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 9$
1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2
3	$a_3 = a_2 + a_1$	3	3

The First Few Values Mod 9

(We skip mod 8 since mod 4 didn't work).
In this section all $\equiv$ are mod 9.

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 9$
1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2
3	$a_3 = a_2 + a_1$	3	3
4	$a_4 = a_3 + a_2$	5	5

The First Few Values Mod 9

(We skip mod 8 since mod 4 didn't work).
In this section all $\equiv$ are mod 9.

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 9$
1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2
3	$a_3 = a_2 + a_1$	3	3
4	$a_4 = a_3 + a_2$	5	5
5	$a_5 = a_4 + a_2$	7	7

The First Few Values Mod 9

(We skip mod 8 since mod 4 didn't work).
In this section all $\equiv$ are mod 9.

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 9$
1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2
3	$a_3 = a_2 + a_1$	3	3
4	$a_4 = a_3 + a_2$	5	5
5	$a_5 = a_4 + a_2$	7	7
6	$a_6 = a_5 + a_3$	10	1

The First Few Values Mod 9

(We skip mod 8 since mod 4 didn't work).
In this section all $\equiv$ are mod 9.

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 9$
1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2
3	$a_3 = a_2 + a_1$	3	3
4	$a_4 = a_3 + a_2$	5	5
5	$a_5 = a_4 + a_2$	7	7
6	$a_6 = a_5 + a_3$	10	1
7	$a_7 = a_6 + a_3$	13	4

The First Few Values Mod 9

(We skip mod 8 since mod 4 didn't work).
In this section all $\equiv$ are mod 9.

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 9$
1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2
3	$a_3 = a_2 + a_1$	3	3
4	$a_4 = a_3 + a_2$	5	5
5	$a_5 = a_4 + a_2$	7	7
6	$a_6 = a_5 + a_3$	10	1
7	$a_7 = a_6 + a_3$	13	4
8	$a_8 = a_7 + a_4$	18	0

The First Few Values Mod 9

(We skip mod 8 since mod 4 didn't work).
In this section all $\equiv$ are mod 9.

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 9$
1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2
3	$a_3 = a_2 + a_1$	3	3
4	$a_4 = a_3 + a_2$	5	5
5	$a_5 = a_4 + a_2$	7	7
6	$a_6 = a_5 + a_3$	10	1
7	$a_7 = a_6 + a_3$	13	4
8	$a_8 = a_7 + a_4$	18	0
9	$a_9 = a_8 + a_4$	23	5

The First Few Values Mod 9

(We skip mod 8 since mod 4 didn't work).
In this section all $\equiv$ are mod 9.

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 9$
1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2
3	$a_3 = a_2 + a_1$	3	3
4	$a_4 = a_3 + a_2$	5	5
5	$a_5 = a_4 + a_2$	7	7
6	$a_6 = a_5 + a_3$	10	1
7	$a_7 = a_6 + a_3$	13	4
8	$a_8 = a_7 + a_4$	18	0
9	$a_9 = a_8 + a_4$	23	5
10	$a_{10} = a_9 + a_5$	30	3

The First Few Values Mod 9

(We skip mod 8 since mod 4 didn't work).
In this section all $\equiv$ are mod 9.

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod 9$
1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2
3	$a_3 = a_2 + a_1$	3	3
4	$a_4 = a_3 + a_2$	5	5
5	$a_5 = a_4 + a_2$	7	7
6	$a_6 = a_5 + a_3$	10	1
7	$a_7 = a_6 + a_3$	13	4
8	$a_8 = a_7 + a_4$	18	0
9	$a_9 = a_8 + a_4$	23	5
10	$a_{10} = a_9 + a_5$	30	3
11	$a_{11} = a_{10} + a_5$	37	1

The First Few Values Mod 9

(We skip mod 8 since mod 4 didn't work).
In this section all $\equiv$ are mod 9.

n	$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$	a_n	$a_n \pmod{9}$
1	a_1	1	1
2	$a_2 = a_1 + a_1$	2	2
3	$a_3 = a_2 + a_1$	3	3
4	$a_4 = a_3 + a_2$	5	5
5	$a_5 = a_4 + a_2$	7	7
6	$a_6 = a_5 + a_3$	10	1
7	$a_7 = a_6 + a_3$	13	4
8	$a_8 = a_7 + a_4$	18	0
9	$a_9 = a_8 + a_4$	23	5
10	$a_{10} = a_9 + a_5$	30	3
11	$a_{11} = a_{10} + a_5$	37	1

No pattern here. But $a_8 \equiv 0 \pmod{9}$.

Lets Try Same Approach as Mod 5

All $\equiv$ are mod 9

Lets Try Same Approach as Mod 5

All $\equiv$ are mod 9

We get the same equation:

$$a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}.$$

Lets Try Same Approach as Mod 5

All $\equiv$ are mod 9

We get the same equation:

$$a_{2m-1} \equiv a_{2m} \equiv a_{2m+1}.$$

WORK IN GROUPS TO GET SOME $a_{m'} \equiv 0$.

Vote on Mod 9

I suspect you did not succeed. **Vote**

Vote on Mod 9

I suspect you did not succeed. **Vote**

1. $(\exists^{\infty} n)[a_n \equiv 0]$ and this has been proven (with a new technique I have not shown yet).

Vote on Mod 9

I suspect you did not succeed. **Vote**

1. $(\exists^\infty n)[a_n \equiv 0]$ and this has been proven (with a new technique I have not shown yet).
2. There is NOT an infinite number of a_n with $a_n \equiv 0$ and this has been proven.

Vote on Mod 9

I suspect you did not succeed. **Vote**

1. $(\exists^\infty n)[a_n \equiv 0]$ and this has been proven (with a new technique I have not shown yet).
2. There is NOT an infinite number of a_n with $a_n \equiv 0$ and this has been proven.
3. The question is **UNKNOWN TO G-K-M**