

START

RECORDING

Sequences, Series and Summation / Product Notation

CMSC 250

Sequences and Series

- A **sequence** is a **function** from the naturals to the complex numbers (but we often use reals).
 - Typical notation: $a: \mathbb{N} \rightarrow \mathbb{C}$
 - Example:
 - 1, 1.5, 2, 2.5, ...

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 - Typical notation: $a: \mathbb{N} \rightarrow \mathbb{C}$
 - Example:
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- There are **three** ways to specify a sequence:
 - Outlining Terms
 - Closed Form
 - Recurrence

Outlining Terms

- Examples:
 - 1, 2, 3, 4, 5, ...

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 - 1, 2, 3, 4, 5, ...
 - 1, 2, 3, 4, 5, 8, 7, 16, 9, 32, ...

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 - 1, 1, 1, 1, ...
 - $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \dots$

Outlining Terms

- Examples:
 - 1, 2, 3, 4, 5, ...
 - 1, 2, 3, 4, 5, 8, 7, 16, 9, 32, ...
 - 1, 1, 1, 1, ...
 - $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \dots$
 - 1, 5, 12, 22, 35, 51, 70, 92, ...

Closed Form

- Examples:
 - $a_n = 2^n, n = 0, 1, 2, \dots$

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 - $c_n = \begin{cases} n, & \text{if } n \text{ is odd} \\ 2^{n/2}, & \text{if } n \text{ is even} \end{cases}$
 - $d_k = k! + k^3, k = 1, 2, 3, \dots$

Recursive Functions

- Examples:

- $$F_n = \begin{cases} 1, & \text{if } n = 0, 1 \\ F_{n-1} + F_{n-2}, & \text{if } n \geq 2 \end{cases}$$

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- $a_1 = 1$
$$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$$
 - Note $\lfloor n/2 \rfloor$ is $n/2$ rounded down.

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- $a_1 = 1$

$$a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$$

- Note $\lfloor n/2 \rfloor$ is $n/2$ rounded down.

- $b_1 = 2$

$$b_2 = 3$$

$$b_n = 2b_{n-1} - b_{n-2}$$

Recursion: Good Idea?

- Example: Fibonacci

$$F_n = \begin{cases} 1, & \text{if } n = 0, 1 \\ F_{n-1} + F_{n-2}, & \text{if } n \geq 2 \end{cases}$$

- We **can** use recursion to compute, say, F_{1000}
- Is it a good idea?

Yes

No

Something
Else

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- Recomputing terms + hidden memory cost of recursion!

Recursion: Done Right

- Is there a better way to compute F_{1000} ?

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Recursion: Done Right

- Is there a better way to compute F_{1000} ?



1. Store the values of $F_0 = 1$, $F_1 = 1$ in an array A.
2. for $i = 2$ to 1000
 $F_i = A[i - 1] + A[i - 2]$
 $A[i] = F_i$
end

- This is a very elementary example of a **very useful technique** called *dynamic programming*.

Closed Formula for Fibonacci

- The closed-form formula for F_n is:

$$F_n = \frac{1}{\sqrt{5}} \underbrace{\left(\frac{1 + \sqrt{5}}{2} \right)^n}_{\phi} - \frac{1}{\sqrt{5}} \underbrace{\left(\frac{1 - \sqrt{5}}{2} \right)^n}_{\psi}$$

- Roughly: $F_n \approx \phi^n \approx (1.618)^n * \frac{1}{\sqrt{5}}$

Recursion vs Closed Formula

1. Computation:

- Recursion leads to a fast dynamic program.
- Classic recursion is elegant.
- Closed form: faster, but numerical issues arise.

2. Rate of growth:

- Recursion gives no hint as to **how big** F_n is.
- Closed form yields $F_n \approx (1.618)^n * \frac{1}{\sqrt{5}}$

Why is Sequence Called The Fibonacci Sequence?

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- So who was the first person to discover what we call the Fib Numbers?

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- Pringas writings from 500 BC have hints of these sequences Sanskrit Poetry.
- See Bill's Blog post for more on the history
 - <https://blog.computationalcomplexity.org/2021/12/did-lane-hemaspaandra-invent-fib-numbers.html>

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5. I am glad I am in Math.

Summation Notation

- Suppose I have some terms of a sequence, let's say $a_1, a_2, a_3, \dots, a_k$.
- Their sum, $a_1 + a_2 + a_3 + \dots + a_k$ is denoted as:

$$\sum_{i=1}^k a_i$$

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The diagram shows the summation notation $\sum_{i=1}^k a_i$. The index k at the top of the sigma symbol is circled in blue, with a blue arrow pointing to it from the text "Index of last term to be included in the sum". The index $i=1$ at the bottom of the sigma symbol is circled in green, with a green arrow pointing to it from the text "Index of term with which the sum begins".

Index of last term to be included in the sum

Index of term with which the sum begins

$$\sum_{i=1}^k a_i$$

Examples

$$\sum_{i=1}^2 a_i = a_1 + a_2$$

$$\sum_{i=1}^1 a_i = a_1$$

$$\sum_{i=1}^0 a_i = ?$$

0

1

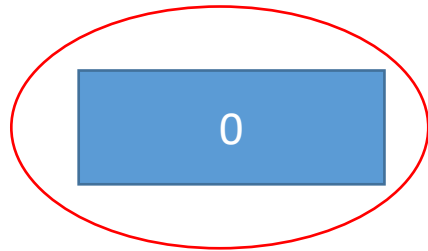
Something
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- Two reasons for this:
 - a) *(Intuitive)* If you add together 0 things, you get 0. Duh.

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b) *(Mathematical)* The following formula should work regardless of our choice of integer variable n_1 :

$$\sum_{i=1}^n a_i = \sum_{i=1}^{n_1} a_i + \sum_{i=n_1+1}^n a_i$$

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So what happens if we pick $n_1 = 0$?

$$\sum_{i=1}^0 a_i = 0$$

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So what happens if we pick $n_1 = 0$?

Then, for this to work, it's necessary that $\sum_{i=1}^0 a_i = 0$

Product Notation

- The **product**, $a_1 \cdot a_2 \cdot \dots \cdot a_k$ is denoted as:

Index of term **with which the product begins**

$$\prod_{i=1}^k a_i$$

Index of last term **to be included** in the product

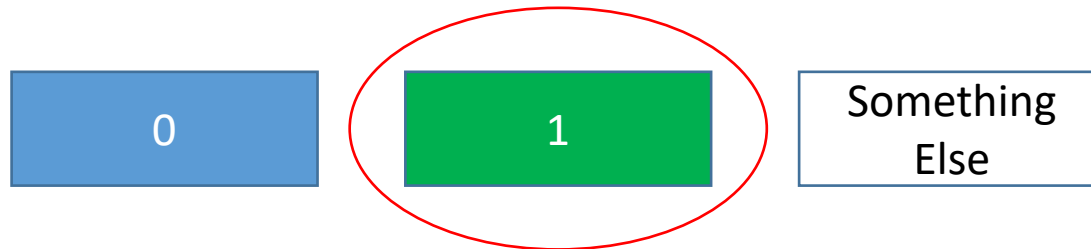
$$\prod_{i=1}^0 a_i = \dots$$

0

1

Something
Else

$$\prod_{i=1}^0 a_i = \dots$$



$$\prod_{i=1}^0 a_i = 1$$

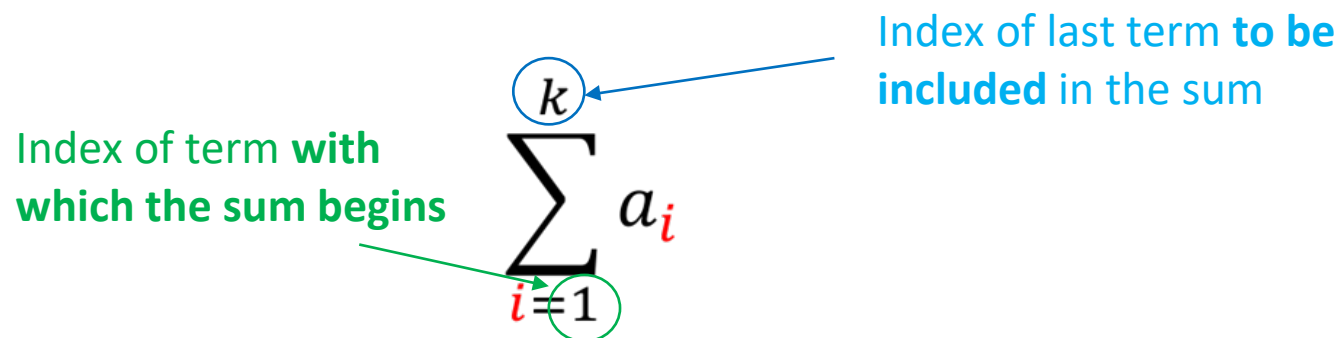
- The following formula has to work for all choices of $n_1 \in \mathbb{N}$:

$$\prod_{i=1}^n a_i = \prod_{i=1}^{n_1} a_i \cdot \prod_{i=n_1+1}^n a_i$$

- So, for $n_1 = 0$, we need $\prod_{i=1}^0 a_i = 1$

Sum / Product Notation

- Suppose I have some terms of a sequence, let's say $a_1, a_2, a_3, \dots, a_k$.
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Index of last term to be included in the sum

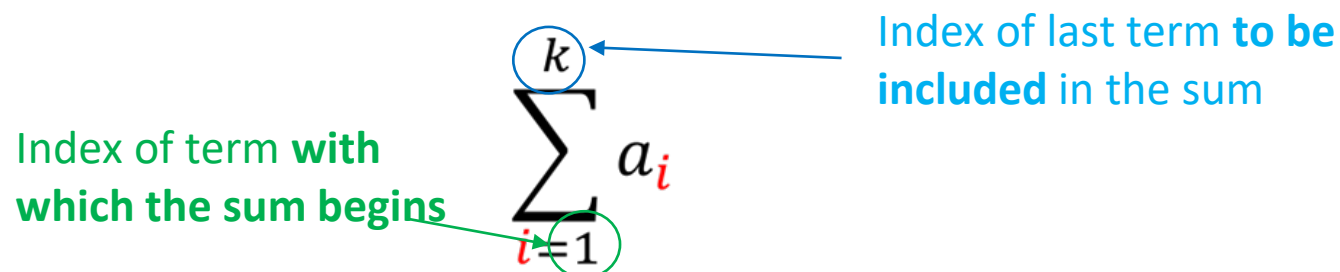
Index of term with which the sum begins

$$\sum_{i=1}^k a_i$$

- Their **product**, $a_1 \cdot a_2 \cdot \dots \cdot a_k$ is denoted as:

Sum / Product Notation

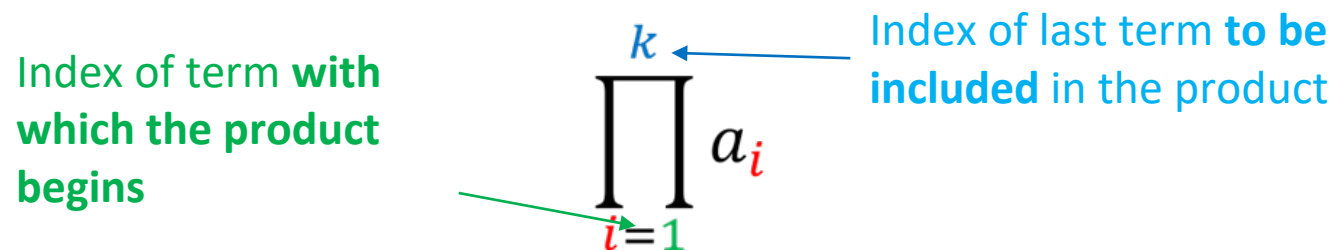
- Suppose I have some terms of a sequence, let's say $a_1, a_2, a_3, \dots, a_k$.
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$$\sum_{i=1}^k a_i$$

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The diagram shows the product notation $\prod_{i=1}^k a_i$. The index k at the top is circled in blue, with a blue arrow pointing to it from the text "Index of last term to be included in the product". The index $i=1$ at the bottom is circled in green, with a green arrow pointing to it from the text "Index of term with which the product begins".

$$\prod_{i=1}^k a_i$$

Sum / Product Notation

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- Their **sum**, $a_1 + a_2 + a_3 + \dots + a_k$ is denoted as:

$$\sum_{r=1}^k a_r$$

“Running” (or “looping” indices can be anything we want! (i, j, k, \dots) **as long as I use the same variable in the Σ and Π symbols and the variable representing the sequence term!**

- Their **product**, $a_1 \cdot a_2 \cdot \dots \cdot a_k$ is denoted as:

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Sum-Product Notation

- We can have certain *exclusionary conditions* under the Σ and Π symbols.
- Examples:

$$\sum_{\substack{m=0 \\ m \text{ even}}}^{100} S_m$$

$$\sum_{\substack{m=0 \\ 3 \mid m}}^{100} S_m \neq \sum_{\substack{m=0 \\ 3 \mid S_m}}^{100} S_m$$

Series and Partial Sums

- A **series** is the **sum** of **all** elements of an **infinite** sequence.

$$\sum_{i=0}^{+\infty} a_i = a_0 + a_1 + a_2 + \dots$$

Or 1, if we start at 1

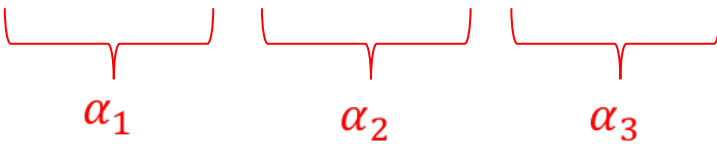
- A **partial sum** of a sequence, denoted S_n , is the sum ranging from the first up to (and including) the n^{th} term of a (usually infinite) sequence:

$$S_n = \sum_{i=0}^n a_i = a_0 + a_1 + a_2 + \dots + a_n$$

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Famous Sequences

- **Arithmetic** (often called the arithmetic **progression**):

$$a, a + d, a + 2d, a + 3d, \dots \text{ where } d \in \mathbb{R}$$


The diagram illustrates the sequence terms $a, a + d, a + 2d, a + 3d, \dots$ with red brackets indicating the differences between consecutive terms. The first bracket is labeled α_1 , the second α_2 , and the third α_3 .

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The diagram shows the sequence terms $a, a + d, a + 2d, a + 3d, \dots$ with red brackets underneath. The first bracket spans from a to $a + d$ and is labeled α_1 . The second bracket spans from $a + d$ to $a + 2d$ and is labeled α_2 . The third bracket spans from $a + 2d$ to $a + 3d$ and is labeled α_3 .

- Question: which among the following is the correct characterization for a_n ?

$$d \cdot a_{n-1}$$

$$\alpha_0 + d \cdot a_{n-1}$$

$$\alpha_0 + n \cdot d$$

$$\alpha_0 + (n - 1) \cdot d$$

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α_1 α_2 α_3

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A Question for You

- **Arithmetic** (often called the arithmetic **progression**):

$$a, a + d, a + 2d, a + 3d, \dots \text{ where } d \in \mathbb{R}$$

- Should we allow $d = 0$?

YES

NO

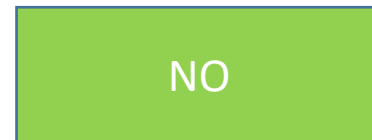
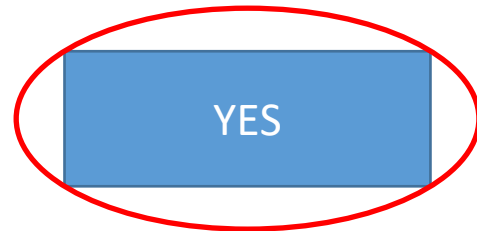
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It will be a pretty boring sequence, but it will still be a sequence!



Famous Sequences

- **Geometric** sequence (or **progression**):

$$a, am, am^2, am^3, \dots \quad m \in \mathbb{R}$$

$$\begin{array}{ccc} \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} \\ \alpha_1 & \alpha_2 & \alpha_3 \end{array}$$

Famous Sequences

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Diagram illustrating the progression a, am, am^2, am^3, \dots with red brackets indicating the common ratio α_1 between a and am , α_2 between am and am^2 , and α_3 between am^2 and am^3 .

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$$(m - 1)^n \cdot a_0$$

$$m^n a_0$$

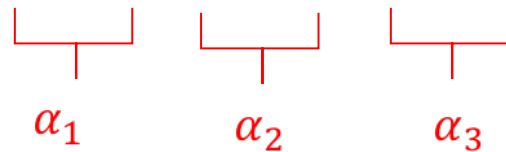
$$m^{n-1} a_0$$

$$m \cdot n \cdot a_0$$

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α_1 α_2 α_3

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$$m^n a_0$$

$$m^{n-1} a_0$$

$$m \cdot n \cdot a_0$$

The Gauss Story



- Gauss was a great mathematician (1777-1855)
- When Gauss was in 1st grade, the class was misbehaving.
- For punishment, the teacher made everyone compute

$$1 + 2 + \dots + 100$$

- Gauss did it in 2 minutes. Can you?

The Gauss Trick

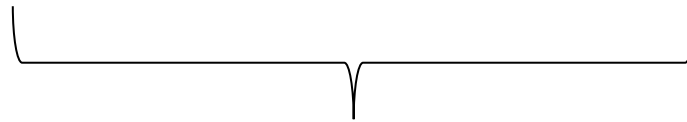


$$S = 1 + 2 + \dots + 100$$

$$S = 100 + 99 + \dots + 1$$

+

$$2S = 101 + 101 + \dots + 101$$



100 terms

$$\Rightarrow 2S = 101 * 100 = 10100 \Rightarrow S = 5050$$

And Now the Rest of the Story



And Now the Rest of the Story



- This is a **complete fabrication!**
- This is how this story has progressed over time:

And Now the Rest of the Story



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And Now the Rest of the Story



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YEAR	GRADE	SERIES
1960	5 th	$1 + 2 + \dots + 60$

And Now the Rest of the Story



- This is a **complete fabrication!**
- This is how this story has progressed over time:

YEAR	GRADE	SERIES
1960	5 th	$1 + 2 + \dots + 60$
1980	3 rd	$1 + 2 + \dots + 80$

And Now the Rest of the Story



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- This is how this story has progressed over time:

YEAR	GRADE	SERIES
1960	5 th	$1 + 2 + \dots + 60$
1980	3 rd	$1 + 2 + \dots + 80$
2000s	1 st	$1 + 2 + \dots + 100$

And Now the Rest of the Story



- This is a **complete fabrication!**
- This is how this story has progressed over time:

YEAR	GRADE	SERIES
1960	5 th	$1 + 2 + \dots + 60$
1980	3 rd	$1 + 2 + \dots + 80$
2000s	1 st	$1 + 2 + \dots + 100$

- The story seems to have converged to 1st grade, $1 + 2 + \dots + 100$

Famous Sequences

- **Harmonic:**

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

- **Fibonacci:** $F_0 = F_1 = 1$ and $\forall n \geq 2, F_n = F_{n-1} + F_{n-2}$

$$1, 1, 2, 3, 5, 8, 13, 21, \dots$$

What We'll Do Next

- We will have an intro to **induction**.
- The following can be proven via induction:

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

STOP

RECORDING