# START RECORDING

# Sequences, Series and Summation / Product Notation

CMSC 250

# Sequences and Series

- A **sequence** is a **function** from the naturals to the complex numbers (but we often use reals).
    - Typical notation: $a\colon \mathbb{N} \to \mathbb{C}$
    - Example:
        - 1, 1.5, 2, 2.5, …

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  - Typical notation: $a\colon\ \mathbb{N} \rightarrow \mathbb{C}$
  - Example:
    - 1, 1.5, 2, 2.5, …
- There are three ways to specify a sequence:
  - Outlining Terms
  - Closed Form
  - Recurrence

# Outlining Terms

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  - $1, 1, 1, 1, \ldots$
  - $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \ldots$

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  - $1, 1, 1, 1, \ldots$
  - $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \ldots$
  - $1, 5, 12, 22, 35, 51, 70, 92, \ldots$

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  - $d_k = k! + k^3,\ k = 1, 2, 3, \ldots$

# Recursive Functions

- Examples:
  - $F_n = \begin{cases} 1, & \text{if } n = 0, 1 \\ F_{n-1} + F_{n-2}, & \text{if } n \geq 2 \end{cases}$

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  - $a_1 = 1$

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  - $b_1 = 2$

$$b_2 = 3$$
$$b_n = 2b_{n-1} - b_{n-2}$$

# Recursion: Good Idea?

- Example: Fibonacci

$$F_n = \begin{cases} 1\,, & if\ n = 0, 1 \\ F_{n-1} + F_{n-2}\,, & if\ n \geq 2 \end{cases}$$

- We ***can*** use recursion to compute, say, $F_{1000}$
- Is it a good idea?

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- Recomputing terms + hidden memory cost of recursion!

# Recursion: Done Right

- Is there a better way to compute $F_{1000}$?

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1. Store the values of $F_0 = 1,\ F_1 = 1$ in an array A.
2. for i = 2 to 1000

   $F_i = A[i-1] + A[i-2]$

   $A[i] = F_i$

   end

- This is a very elementary example of a **very useful technique** called ***dynamic programming.***

# Closed Formula for Fibonacci

- The closed-form formula for $F_n$ is:

$$F_n = \frac{1}{\sqrt{5}} \underbrace{\left(\frac{1+\sqrt{5}}{2}\right)}_{\phi}^{n} - \frac{1}{\sqrt{5}} \underbrace{\left(\frac{1-\sqrt{5}}{2}\right)}_{\psi}^{n}$$

- Roughly: $F_n \approx \phi^n \approx (1.618)^n * \frac{1}{\sqrt{5}}$

# Recursion vs Closed Formula

1. Computation:
   - Recursion leads to a fast dynamic program.
   - Classic recursion is elegant.
   - Closed form: faster, but numerical issues arise.
2. Rate of growth:
   - Recursion gives no hint as to **how big** $F_n$ is.
   - Closed form yields $F_n \approx (1.618)^n * \frac{1}{\sqrt{5}}$

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  - Note that Columbus was the last person to discover America.
- So who was the first person to discover what we call the Fib Numbers?

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- Pringas writings from 500 BC have hints of these sequences .... Sanskrit Poetry.
- See Bill's Blog post for more on the history
  - https://blog.computationalcomplexity.org/2021/12/did-lane-hemaspaandra-invent-fib-numbers.html

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5. I am glad I am in Math.

# Summation Notation

- Suppose I have some terms of a sequence, let's say $a_1, a_2, a_3, \ldots, a_k$.
- Their sum, $a_1 + a_2 + a_3 + \cdots + a_k$ is denoted as:

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Index of term **with which the sum begins**

# Examples

$$\sum_{i=1}^{2} a_i = a_1 + a_2$$

$$\sum_{i=1}^{1} a_i = a_1$$

$$\sum_{i=1}^{0} a_i = ?$$

0

1

Something Else

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$$\sum_{i=1}^{0} a_i = 0$$

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  - *a)* *(Intuitive)* If you add together 0 things, you get 0. Duh.

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So what happens if we pick $n_1 = 0$?

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So what happens if we pick $n_1 = 0$?
Then, for this to work, it's necessary that $\sum_{i=1}^{0} a_i = 0$

# Product Notation

- The **product**, $a_1 \cdot a_2 \cdot \ldots \cdot a_k$ is denoted as:

$$\prod_{i=1}^{k} a_i$$

Index of term **with which the product begins**

Index of last term **to be included** in the product

$$\prod_{i=1}^{0} a_i = \cdots$$

0

1

Something Else

$$\prod_{i=1}^{0} a_i = \cdots$$

0

1

Something Else

$$\prod_{i=1}^{0} a_i = 1$$

- The following formula has to work for all choices of $n_1 \in \mathbb{N}$:

$$\prod_{i=1}^{n} a_i = \prod_{i=1}^{n_1} a_i \cdot \prod_{i=n_1+1}^{n} a_i$$

- So, for $n_1 = 0$, we need $\prod_{i=1}^{0} a_i = 1$

# Sum / Product Notation

- Suppose I have some terms of a sequence, let's say $a_1, a_2, a_3, \ldots, a_k$.
- Their **sum**, $a_1 + a_2 + a_3 + \cdots + a_k$ is denoted as:

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- Their **sum**, $a_1 + a_2 + a_3 + \cdots + a_k$ is denoted as:

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- Their **product**, $a_1 \cdot a_2 \cdot \ldots \cdot a_k$ is denoted as:

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"Running" (or "looping" indices can be anything we want! $(i, j, k, \ldots)$ **as long as I use the same variable in the $\Sigma$ and $\Pi$ symbols and the variable representing the sequence term!**

# Sum-Product Notation

- We can have certain ***exclusionary conditions*** under the $\Sigma$ and $\Pi$ symbols.
- Examples:

$$\sum_{\substack{m=0 \\ m\ even}}^{100} s_m$$

$$\sum_{\substack{m=0 \\ 3\ |\ m}}^{100} s_m \neq \sum_{\substack{m=0 \\ 3\ |\ s_m}}^{100} s_m$$

# Series and Partial Sums

- A ***series*** is the **sum** of **all** elements of an **infinite** sequence.

$$\sum_{i=0}^{+\infty} a_i = a_0 + a_1 + a_2 + \cdots$$

Or 1, if we start at 1

- A **partial sum** of a sequence, denoted $\boldsymbol{S_n}$, is the sum ranging from the first up to (and including) the $n^{th}$ term of a (usually infinite) sequence:

$$S_n = \sum_{i=0}^{n} a_i = a_0 + a_1 + a_2 + \cdots + a_n$$

Or 1, if we start at 1

# Famous Sequences

- **Arithmetic** (often called the arithmetic **progression**):

$$\underbrace{a, a + d}_{\alpha_1}, \underbrace{a + 2d}_{\alpha_2}, \underbrace{a + 3d}_{\alpha_3}, \ldots \text{ where } d \in \mathbb{R}$$

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$\alpha_1$ $\alpha_2$ $\alpha_3$

- Question: which among the following is the correct characterization for $a_n$?

$d \cdot a_{n-1}$

$\alpha_0 + d \cdot a_{n-1}$

$\alpha_0 + n \cdot d$

$\alpha_0 + (n-1) \cdot d$

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# A Question for You

- **Arithmetic** (often called the arithmetic **progression**):

$$a, a + d, a + 2d, a + 3d, \ldots \text{ where } d \in \mathbb{R}$$

- Should we allow $d = 0$?

YES

NO

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- Should we allow $d = 0$?

It will be a pretty boring sequence, but it will still be a sequence!

YES

NO

# Famous Sequences

- **Geometric** sequence (or **progression**):

$$a\ ,\underbrace{am}_{\alpha_1}\ ,\underbrace{am^2}_{\alpha_2}\ ,\underbrace{am^3}_{\alpha_3}\ ,\ldots \qquad m \in \mathbb{R}$$

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| | | | |
|---|---|---|---|
| $(m - 1)^n \cdot a_0$ | $m^n a_0$ | $m^{n-1} a_0$ | $m \cdot n \cdot a_0$ |

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# The Gauss Story

- Gauss was a great mathematician (1777-1855)
- When Gauss was in 1st grade, the class was misbehaving.
- For punishment, the teacher made everyone compute

$$1 + 2 + \cdots + 100$$

- Gauss did it in 2 minutes. Can you?

# The Gauss Trick

$$S = 1 + 2 + \cdots + 100$$
$$S = 100 + 99 + \cdots + 1$$

$+$

$$2S = \underbrace{101 + 101 + \cdots + 101}_{\text{100 terms}}$$

$$\Rightarrow 2S = 101 * 100 = 10100 \Rightarrow S = 5050$$

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|---|---|---|
| 1960 | 5th | 1 + 2 + … + 60 |
| 1980 | 3rd | 1 + 2 + … + 80 |

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| 1980 | $3^{rd}$ | 1 + 2 + … + 80 |
| 2000s | $1^{st}$ | 1 + 2 + … + 100 |

# And Now the Rest of the Story

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| 1960 | $5^{th}$ | 1 + 2 + ... + 60 |
| 1980 | $3^{rd}$ | 1 + 2 + ... + 80 |
| 2000s | $1^{st}$ | 1 + 2 + ... + 100 |

- The story seems to have converged to $1^{st}$ grade, 1 + 2 + ... + 100

# Famous Sequences

- **Harmonic**:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

- **Fibonacci**: $F_0 = F_1 = 1$ and $\forall n \geq 2, F_n = F_{n-1} + F_{n-2}$

1, 1, 2, 3, 5, 8, 13, 21, ...

# What We'll Do Next

- We will have an intro to **induction.**
- The following can be proven via induction:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

STOP RECORDING