Untimed Midterm Two Solutions

An Interesting Sum

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May use $(n-1)^{11} \sim n^{11} - 11n^{10}$.
BY CONSTRUCTIVE INDUCTION find A such that

$$(\forall n \geq 100)\left[\sum_{i=100}^{n} i^{10} \leq An^{11}\right].$$

Base Case

IB $n = 100$. $\sum_{i=100}^{100} i^{10} = 100^{10}$.

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$$A \geq \frac{100^{10}}{100^{11}} = \frac{1}{100}.$$

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$$100^{10} \leq A \times 100^{11}.$$

$$A \geq \frac{100^{10}}{100^{11}} = \frac{1}{100}.$$

So the constraint is $A \geq \frac{1}{100}$.

IH and IS

IH $\sum_{i=100}^{n-1} i^{10} \le A(n-1)^{11}$.

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IH $\sum_{i=100}^{n-1} i^{10} \le A(n-1)^{11}$.

IS

$$\sum_{i=100}^{n} i^{10} = (\sum_{i=100}^{n-1} i^{10}) + n^{10} \le A(n-1)^{11} + n^{10}.$$

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$$A(n-1)^{11} + n^{10} \leq An^{11}$$

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We need

$$A(n-1)^{11} + n^{10} \leq An^{11}$$

$$n^{10} \leq An^{11} - A(n-1)^{11} \sim An^{11} - A(n^{11} - 11n^{10})$$

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$$n^{10} \leq An^{11} - An^{11} + 11An^{10} = 11An^{10}$$

$$A \geq \frac{1}{11}.$$

Picking A

The two constraints on A are

1. $A \geq \frac{1}{100}$, and
2. $A \geq \frac{1}{11}$.

Hence we choose $A = \frac{1}{11}$.

Generalization of an Interesting Sum

Generalization of Interesting Sum

May use $(n-1)^a \sim n^a - an^{a-1}$.
BY CONSTRUCTIVE INDUCTION find a constant B such that

$$(\forall n \geq 100)\left[\sum_{i=100}^{n} i^a \leq Bn^{a+1}\right].$$

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IB $n = 100$. $\sum_{i=100}^{100} i^a = (100)^a$. We need that

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IB $n = 100$. $\sum_{i=100}^{100} i^a = (100)^a$. We need that

$$100^a \leq B \times 100^{a+1}.$$

So the constraint is

$$B \geq \frac{100^a}{100^{a+1}} = \frac{1}{100}.$$

IH and IS

IH: $\sum_{i=100}^{n-1} i^a \leq B(n-1)^{a+1}$.

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IH: $\sum_{i=100}^{n-1} i^a \leq B(n-1)^{a+1}$.

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$$\sum_{i=100}^{n} i^a = \sum_{i=100}^{n-1} i^a + n^a \leq B(n-1)^{a+1} + n^a.$$

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IH: $\sum_{i=100}^{n-1} i^a \leq B(n-1)^{a+1}$.

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We need

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IH: $\sum_{i=100}^{n-1} i^a \le B(n-1)^{a+1}$.
IS:

$$\sum_{i=100}^{n} i^a = \sum_{i=100}^{n-1} i^a + n^a \le B(n-1)^{a+1} + n^a.$$

We need

$$B(n-1)^{a+1} + n^a \le Bn^{a+1}$$

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IH: $\sum_{i=100}^{n-1} i^a \leq B(n-1)^{a+1}$.

IS:

$$\sum_{i=100}^{n} i^a = \sum_{i=100}^{n-1} i^a + n^a \leq B(n-1)^{a+1} + n^a.$$

We need

$$B(n-1)^{a+1} + n^a \leq Bn^{a+1}$$

$$n^a \leq Bn^{a+1} - B(n-1)^{a+1} \sim Bn^{a+1} - B(n^{a+1} - (a+1)n^a).$$

IH and IS

IH: $\sum_{i=100}^{n-1} i^a \le B(n-1)^{a+1}$.
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$$\sum_{i=100}^{n} i^a = \sum_{i=100}^{n-1} i^a + n^a \le B(n-1)^{a+1} + n^a.$$

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$$B(n-1)^{a+1} + n^a \le Bn^{a+1}$$

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$$n^a \le Bn^{a+1} - Bn^{a+1} + (a+1)Bn^a) = (a+1)Bn^a$$

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$$n^a \le Bn^{a+1} - Bn^{a+1} + (a+1)Bn^a) = (a+1)Bn^a$$

$$B \ge \frac{1}{a+1}.$$

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Hence pick

$$B = \max\left\{\frac{1}{100}, \frac{1}{a+1}\right\}.$$

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- Prove the above once you found it.

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Problem Find C such that

- $C - 1$ cannot be written as $10x + 13y$ where $x, y \in \mathbb{N}$, and
- $(\forall n \geq C)(\exists x, y \in \mathbb{N})[n = 10x + 13y]$.
- Prove the above once you found it.

We asked you to do it by computer
We will do it today by constructive induction.

Coin Problem Solution. Plan and Base Case

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Plan

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1. If there is a 10-coin then we will swap it out and put in a 13. So we will go $P(n) \rightarrow P(n+3)$. Hence we need for a base case $P(C)$, $P(C+1)$, $P(C+2)$.

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2. If there are no 10 coins then we plan to swap out nine 13-coins (117) and put in twelve 10-coins (120) Hence we need for a base case $P(C)$, $P(C+1)$, $P(C+2)$.

Coin Problem Solution. Plan and Base Case

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IB C, $C+1$, $C+2$ are all of the form $10x+13y$.

IH and IS

IH For all $C \leq n' < n$ there exists x', y' such that $n' = 10x' + 13y'$.

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Case 1 If $x' \ge 1$ then we swap out a 10 and put in a 13.

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Case 1 If $x' \geq 1$ then we swap out a 10 and put in a 13.

$$10(x'-1) + 13(y'+1) = 10x' + 13y' + 3 = n - 3 + 3 = n$$

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Case 2 If $y' \ge 9$ then we swap out 9 13's and put in a 12 10's:

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Case 1 If $x' \ge 1$ then we swap out a 10 and put in a 13.

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Case 2 If $y' \ge 9$ then we swap out 9 13's and put in a 12 10's:

$$10(x'+12) + 13(y'-9) = 10x' + 13y' + 120 - 117 = n - 3 + 3 = n.$$

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Case 3 $x' \le 0$ and $y' \le 8$. Then
$n - 3 = 10x' + 13y' \le 13 \times 8 = 104$.

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Note Since the IB was $C, C+1, C+2$ we have: the theorem holds for $n-3$. So $(\exists x', y)[n-3 = 10x' + 13y']$.
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Case 3 $x' \le 0$ and $y' \le 8$. Then
$n - 3 = 10x' + 13y' \le 13 \times 8 = 104$.
$n \le 107$.

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IH For all $C \le n' < n$ there exists x', y' such that $n' = 10x' + 13y'$.
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Case 1 If $x' \ge 1$ then we swap out a 10 and put in a 13.

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Case 2 If $y' \ge 9$ then we swap out 9 13's and put in a 12 10's:

$$10(x'+12) + 13(y'-9) = 10x' + 13y' + 120 - 117 = n-3+3 = n.$$

Case 3 $x' \le 0$ and $y' \le 8$. Then
$n-3 = 10x' + 13y' \le 13 \times 8 = 104$.
$n \le 107$.
The proof that $P(n-3) \rightarrow P(n)$ only works when $n \ge 108$.

Our Guess and Our Plan to Find C

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We **guess** that the following is true:

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1. 107 **is not** of the form $10x + 13y$.

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1. 107 **is not** of the form $10x + 13y$.
2. 108, 109, 110 **are** of the form $10x + 13y$.

Our Guess and Our Plan to Find C

We **guess** that the following is true:

1. 107 **is not** of the form $10x + 13y$.
2. 108, 109, 110 **are** of the form $10x + 13y$.

What might happen? Cases.

Our Guess and Our Plan to Find C

We **guess** that the following is true:

1. 107 **is not** of the form $10x + 13y$.
2. 108, 109, 110 **are** of the form $10x + 13y$.

What might happen? Cases.

1. 107 is not of the form but 108, 109, 110 are. Then $C = 108$.

Our Guess and Our Plan to Find C

We **guess** that the following is true:

1. 107 **is not** of the form $10x + 13y$.
2. 108, 109, 110 **are** of the form $10x + 13y$.

What might happen? Cases.

1. 107 is not of the form but 108, 109, 110 are. Then $C = 108$.
2. At least one of 108, 109, 110 are not of the form. Find $C \geq 108$ such that $C - 1$ is not of the form but $C, C + 1, C + 2$ are of the form. Thats your C.

Our Guess and Our Plan to Find C

We **guess** that the following is true:

1. 107 **is not** of the form $10x + 13y$.
2. 108, 109, 110 **are** of the form $10x + 13y$.

What might happen? Cases.

1. 107 is not of the form but 108, 109, 110 are. Then $C = 108$.
2. At least one of 108, 109, 110 are not of the form. Find $C \geq 108$ such that $C - 1$ is not of the form but $C, C + 1, C + 2$ are of the form. Thats your C.
3. 107,108,109,110 are of the form. Hence all $n \geq 107$ are of the form.

Our Guess and Our Plan to Find C

We **guess** that the following is true:

1. 107 **is not** of the form $10x + 13y$.
2. 108, 109, 110 **are** of the form $10x + 13y$.

What might happen? Cases.

1. 107 is not of the form but 108, 109, 110 are. Then $C = 108$.
2. At least one of 108, 109, 110 are not of the form. Find $C \geq 108$ such that $C - 1$ is not of the form but $C, C+1, C+2$ are of the form. Thats your C.
3. 107,108,109,110 are of the form. Hence all $n \geq 107$ are of the form.
   Look at 106, 105, ... until you find a number NOT of that form. That number is your $C - 1$ so one more is your C.

107

Assume, BWOC, that there exists $x, y \geq 0$ such that

$$107 = 10x + 13y$$

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Take both sides mod 10 to get
$7 \equiv 3y \mod 10$.

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Assume, BWOC, that there exists $x, y \geq 0$ such that

$$107 = 10x + 13y$$

Take both sides mod 10 to get
$7 \equiv 3y \mod 10$.
If $y \equiv 0, 2, 4, 6, 8$ then $3y$ is even so not $\equiv 7 \pmod{10}$.

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Assume, BWOC, that there exists $x, y \geq 0$ such that

$$107 = 10x + 13y$$

Take both sides mod 10 to get
$7 \equiv 3y \mod 10$.
If $y \equiv 0, 2, 4, 6, 8$ then $3y$ is even so not $\equiv 7 \pmod{10}$.
$y \equiv 1$: $3 \times 1 \equiv 3 \not\equiv 7$.

107

Assume, BWOC, that there exists $x, y \geq 0$ such that

$$107 = 10x + 13y$$

Take both sides mod 10 to get
$7 \equiv 3y \mod 10$.
If $y \equiv 0, 2, 4, 6, 8$ then $3y$ is even so not $\equiv 7 \pmod{10}$.
$y \equiv 1$: $3 \times 1 \equiv 3 \not\equiv 7$.
$y \equiv 3$: $3 \times 3 \equiv 9 \not\equiv 7$.

107

Assume, BWOC, that there exists $x, y \geq 0$ such that

$$107 = 10x + 13y$$

Take both sides mod 10 to get
$7 \equiv 3y \mod 10$.
If $y \equiv 0, 2, 4, 6, 8$ then $3y$ is even so not $\equiv 7 \pmod{10}$.
$y \equiv 1$: $3 \times 1 \equiv 3 \not\equiv 7$.
$y \equiv 3$: $3 \times 3 \equiv 9 \not\equiv 7$.
$y \equiv 5$: $3 \times 5 \equiv 6 \not\equiv 7$.

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Assume, BWOC, that there exists $x, y \geq 0$ such that

$$107 = 10x + 13y$$

Take both sides mod 10 to get
$7 \equiv 3y \mod 10$.
If $y \equiv 0, 2, 4, 6, 8$ then $3y$ is even so not $\equiv 7 \pmod{10}$.
$y \equiv 1$: $3 \times 1 \equiv 3 \not\equiv 7$.
$y \equiv 3$: $3 \times 3 \equiv 9 \not\equiv 7$.
$y \equiv 5$: $3 \times 5 \equiv 6 \not\equiv 7$.
$y \equiv 7$: $3 \times 7 \equiv 1 \not\equiv 7$.

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Assume, BWOC, that there exists $x, y \geq 0$ such that

$$107 = 10x + 13y$$

Take both sides mod 10 to get
$7 \equiv 3y \mod 10$.
If $y \equiv 0, 2, 4, 6, 8$ then $3y$ is even so not $\equiv 7 \pmod{10}$.
$y \equiv 1$: $3 \times 1 \equiv 3 \not\equiv 7$.
$y \equiv 3$: $3 \times 3 \equiv 9 \not\equiv 7$.
$y \equiv 5$: $3 \times 5 \equiv 6 \not\equiv 7$.
$y \equiv 7$: $3 \times 7 \equiv 1 \not\equiv 7$.
$y \equiv 9$: $3 \times 9 \equiv 7$.

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Assume, BWOC, that there exists $x, y \geq 0$ such that

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Take both sides mod 10 to get
$7 \equiv 3y \mod 10$.
If $y \equiv 0, 2, 4, 6, 8$ then $3y$ is even so not $\equiv 7 \pmod{10}$.
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$y \equiv 3$: $3 \times 3 \equiv 9 \not\equiv 7$.
$y \equiv 5$: $3 \times 5 \equiv 6 \not\equiv 7$.
$y \equiv 7$: $3 \times 7 \equiv 1 \not\equiv 7$.
$y \equiv 9$: $3 \times 9 \equiv 7$.
Hence $y \equiv 9 \pmod{10}$. Hence $y \geq 9$.

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Assume, BWOC, that there exists $x, y \geq 0$ such that

$$107 = 10x + 13y$$

Take both sides mod 10 to get
$7 \equiv 3y \mod 10$.
If $y \equiv 0, 2, 4, 6, 8$ then $3y$ is even so not $\equiv 7 \pmod{10}$.
$y \equiv 1$: $3 \times 1 \equiv 3 \not\equiv 7$.
$y \equiv 3$: $3 \times 3 \equiv 9 \not\equiv 7$.
$y \equiv 5$: $3 \times 5 \equiv 6 \not\equiv 7$.
$y \equiv 7$: $3 \times 7 \equiv 1 \not\equiv 7$.
$y \equiv 9$: $3 \times 9 \equiv 7$.
Hence $y \equiv 9 \pmod{10}$. Hence $y \geq 9$.
But $13 \times 9 = 117 > 107$.

107

Assume, BWOC, that there exists $x, y \geq 0$ such that

$$107 = 10x + 13y$$

Take both sides mod 10 to get
$7 \equiv 3y \mod 10$.
If $y \equiv 0, 2, 4, 6, 8$ then $3y$ is even so not $\equiv 7 \pmod{10}$.
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$y \equiv 3$: $3 \times 3 \equiv 9 \not\equiv 7$.
$y \equiv 5$: $3 \times 5 \equiv 6 \not\equiv 7$.
$y \equiv 7$: $3 \times 7 \equiv 1 \not\equiv 7$.
$y \equiv 9$: $3 \times 9 \equiv 7$.
Hence $y \equiv 9 \pmod{10}$. Hence $y \geq 9$.
But $13 \times 9 = 117 > 107$.
Hence no $y \equiv 9 \pmod{10}$ can work.

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Assume, BWOC, that there exists $x, y \geq 0$ such that

$$107 = 10x + 13y$$

Take both sides mod 10 to get
$7 \equiv 3y \mod 10$.
If $y \equiv 0, 2, 4, 6, 8$ then $3y$ is even so not $\equiv 7 \pmod{10}$.
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$y \equiv 3$: $3 \times 3 \equiv 9 \not\equiv 7$.
$y \equiv 5$: $3 \times 5 \equiv 6 \not\equiv 7$.
$y \equiv 7$: $3 \times 7 \equiv 1 \not\equiv 7$.
$y \equiv 9$: $3 \times 9 \equiv 7$.
Hence $y \equiv 9 \pmod{10}$. Hence $y \geq 9$.
But $13 \times 9 = 117 > 107$.
Hence no $y \equiv 9 \pmod{10}$ can work.
Hence no y can work.

108,109,110

Need that 108, 109, 110 ARE of the form $10x + 13y$.

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1. $108 = 3 \times 10 + 6 \times 13$.

108,109,110

Need that 108, 109, 110 ARE of the form $10x + 13y$.

1. $108 = 3 \times 10 + 6 \times 13$.
2. $109 = 7 \times 10 + 3 \times 13$.

108,109,110

Need that 108, 109, 110 ARE of the form $10x + 13y$.

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So we are done! $C = 108$.

Sum of Squares

Fourth Powers Mod 16

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$$X = \{x^4 \pmod{16} : x \in \{0, \dots, 15\}\}.$$

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$X = \{0, 1\}$.

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Contradiction.

$x \equiv 1 \pmod{2} \rightarrow x^4 \equiv 1 \pmod{16}$

We give two proofs.

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Pf One $x = 2k + 1$

$$x^2 = (2k+1)^4 = (2k)^4 + 4(2k)^3 + 6(2k)^2 + 4(2k) + 1^4$$

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Pf Two $x \equiv 1 \pmod 2 \rightarrow x \equiv 1, 3, 5, 7, 9, 11, 13, 15 \pmod{16}$.
We did this earlier.

$x_1^4 + \cdots + x_{14}^4 \equiv 0 \pmod{16} \rightarrow (\forall i)[x_i \equiv 0 \pmod{2})$

Assume that m of the x_i's are odd and $14 - m$ are even.

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But also

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So $m = 0$.

Main Thm and IB

Thm Let $n \geq 0$. Let $k \in \mathbb{N}$. Then $16^n(16k+15)$ cannot be written as the sum of 14 4th powers.

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Prove is by induction on n.
IB $n = 0$. $(\forall k)[16k + 15$ is not the sum of 14 4th powers$]$.
This was proven in an earlier part.

IH and IS

IH For all k', $16^{n-1}(16k' + 15)$ is not the sum of 14 4th powers.
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So $x_1^4 + \cdots + x_{14}^4 \equiv 0 \pmod{16}$.

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Assume, BWOC: $(\exists x_1, \ldots, x_{14})[16^n(16k+15) = x_1^4 + \cdots + x_{14}^4]$.
So $x_1^4 + \cdots + x_{14}^4 \equiv 0 \pmod{16}$.
By earlier part $x_1, \ldots, x_{14} = 2y_1, \ldots, 2y_{14}$.

IH and IS (cont)

$$(\exists x_1, \ldots, x_{14})[16^n(16k+15) = x_1^4 + \cdots + x_{14}^4].$$

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This is a contradiction.