

START

RECORDING

Mathematical Induction: Introduction and Basic Problems

CMSC 250

INTRO AND BASIC SEQUENCE PROBLEMS

The Idea Behind Induction

- Suppose that we want to prove that a proposition $P(n)$ is true for all natural numbers n .

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- Suppose that we want to prove that a proposition $P(n)$ is true for all natural numbers n .
- We will prove **two** separate things:
 1. For $n = 0$, $P(n)$ is true (*simplifiable to “ $P(0)$ is true”*).

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- Suppose that we want to prove that a proposition $P(n)$ is true for all natural numbers n .
- We will prove **two** separate things:
 1. $P(0)$ is true.
 2. For all $n \geq 1$, $P(n) \Rightarrow P(n + 1)$

The Induction Principle

- From
 - Base Case (BC): $P(0)$
 - Induction Step (IS): $\forall n \geq 0, P(n) \implies P(n + 1)$
- We can deduce $\forall n \geq 0, P(n)$.

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- Why does the Induction Principle Work?

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- We can deduce $\forall n \geq 0, P(n)$.
- Why does the Induction Principle Work?
- Lets say you have the BC and the IS. You want to know if $P(17)$ is true.
- You have
 - $P(0)$
 - $P(0) \Rightarrow P(1)$
 - $P(1) \Rightarrow P(2)$
 - \vdots
 - $P(16) \Rightarrow P(17)$
- Hence you have $P(17)$

More Succinctly

- If you have
 - BC: $P(0)$
 - IS: $\forall n \geq 0, P(n) \implies P(n + 1)$
- Then for any $n \geq 0$, one can construct a proof of $P(n)$.
- Hence for any $n \geq 0$, $P(n)$ is true.

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How We'll Make It Work

1. **Inductive base:** We will prove (explicitly, no matter how dumb it may sometimes seem) that $P(0)$ is true
 2. **Inductive hypothesis:** We will assume that, for $n \geq 0$, $P(n)$ holds.
 3. **Inductive step:** We will prove that if $P(n)$ holds, then $P(n + 1)$ holds.
- So everything falls into place!

SUM PROBLEMS

$$\sum_{i=0}^n f(n)$$

The Gaussian Sum

- We will prove that the sum of the first n numbers is equal to $\frac{n(n+1)}{2}$.
- Symbolically:

$$1 + 2 + 3 + \cdots + (n - 1) + n = \frac{n(n + 1)}{2}$$
$$\sum_{i=1}^n i = \frac{n(n + 1)}{2}$$

Inductive Base

- For $n = 0$, we will **prove** that $P(0)$ holds

$$\sum_{i=1}^0 i = \frac{0(0+1)}{2}$$

Remember: $P(n)$ is

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

- **LHS:** $\sum_{i=1}^0 i = 0$ (recall this fact from our sequences lecture)
- **RHS:** $\frac{0(0+1)}{2} = 0$
- Since LHS = RHS for $n = 0$, $P(0)$ has been proven true.

Inductive Hypothesis

- For $n \geq 0$, we **assume** that $P(n)$ is true:

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

So, we **assume** that

$$P(n) \Leftrightarrow \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

is true for an $n \geq 0$

- Inductive Hypothesis done!

Inductive Step

- Given that $P(n)$ is true, we will **prove** that $P(n + 1)$ is true.

$$\sum_{i=1}^n i = \frac{n(n+1)}{2} \Rightarrow \sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

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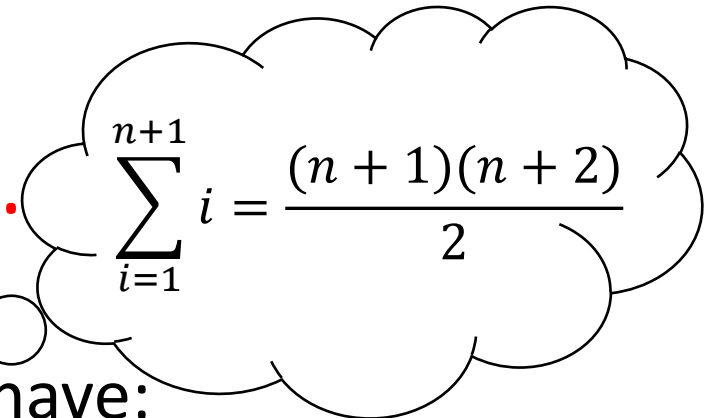
Inductive Step

- Given that $P(n)$ is true, we will **prove** that $P(n + 1)$ is true.

$$\sum_{i=1}^n i = \frac{n(n+1)}{2} \Rightarrow \sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

This is our goal!

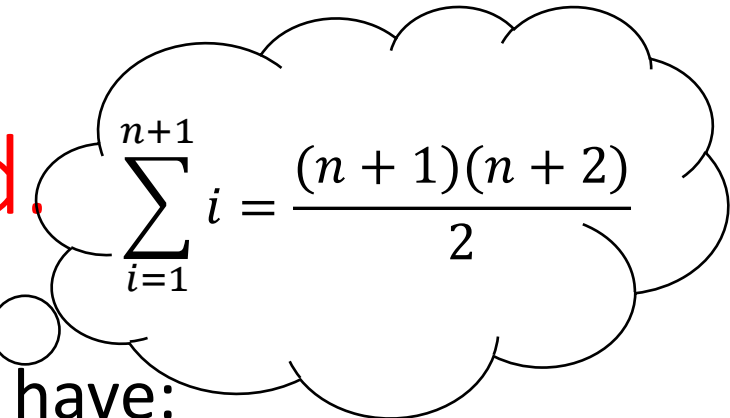
Inductive Step, contd.


$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

- Starting from the **LHS** of the relation to prove, we have:

$$\sum_{i=1}^{n+1} i = 1 + 2 + \cdots + n + (n + 1)$$

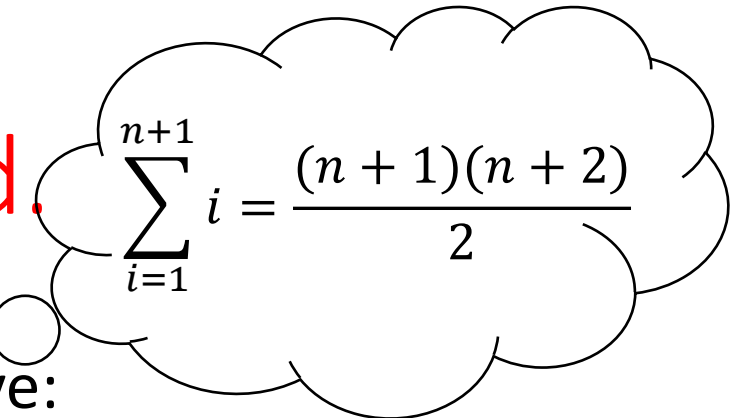
Inductive Step, contd.


$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

- Starting from the **LHS** of the relation to prove, we have:

$$\sum_{i=1}^{n+1} i = 1 + 2 + \cdots + n + (n+1) = \sum_{i=1}^n i + (n+1) \quad (1)$$

Inductive Step, contd.


$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

- Starting from the **LHS** of the relation to prove, we have:

$$\sum_{i=1}^{n+1} i = 1 + 2 + \cdots + n + (n+1) = \sum_{i=1}^n i + (n+1) \quad (1)$$

- **From the Inductive Hypothesis**, we have that

$$\sum_{i=1}^n i = \frac{n(n+1)}{2} \quad (2)$$

Inductive Step, contd.

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

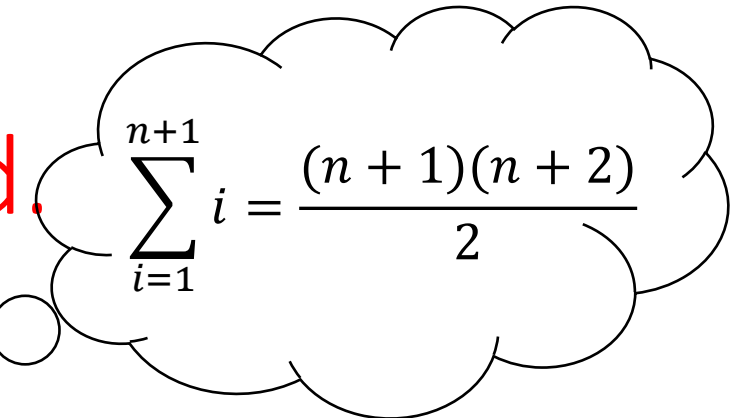
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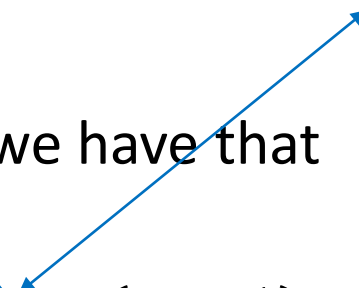
Inductive Step, contd.


$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

- Starting from the **LHS** of the relation to prove, we have:

$$\sum_{i=1}^{n+1} i = 1 + 2 + \dots + n + (n+1) = \sum_{i=1}^n i + (n+1)(1)$$

- From the Inductive Hypothesis**, we have that


$$\sum_{i=1}^n i = \frac{n(n+1)}{2} \quad (2)$$

- Substituting (2) into (1) yields (next slide):

Inductive Step, contd.

$$\sum_{i=1}^{n+1} i = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{(n+2)(n+1)}{2} = \text{RHS}$$

Inductive Step, contd.

$$\sum_{i=1}^{n+1} i = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{(n+2)(n+1)}{2} = \text{RHS}$$

- So, when $P(n)$ is true, $P(n+1)$ was also proven true.
- We conclude that $P(n)$ is true $\forall n \geq 0$.
- WE ARE DONE.

Here's Another!

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

Inductive Base

- For $n = 0$, $\text{LHS} = \sum_{i=1}^0 i^2 = 0$
- $\text{RHS} = \frac{0(0+1)(2*0+1)}{2} = 0$
- Since $\text{LHS} = \text{RHS}$, $P(0)$ holds and we are done.

Inductive Base

- For $n = 0$, $\text{LHS} = \sum_{i=1}^0 i^2 = 0$
- $\text{RHS} = \frac{0(0+1)(2*0+1)}{2} = 0$
- Since $\text{LHS} = \text{RHS}$, $P(0)$ holds and we are done.
- You could also start from $n = 1!$ LHS = RHS in both cases
 - $n = 0$ sometimes makes the math easier (RHS in this case)

Inductive Hypothesis

- Suppose that $n \geq 0$. (Or 1 in the alternative scenario)
- We will then assume $P(n)$, i.e:

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

Inductive Step

- We will now attempt to prove $P(n + 1)$, i.e

$$\sum_{i=1}^{n+1} i^2 = \frac{(n + 1)(n + 2)(2n + 3)}{6}$$

Careful with
factoring please!!!

Inductive Step

- We will now attempt to prove $P(n + 1)$, i.e

$$\sum_{i=1}^{n+1} i^2 = \frac{(n + 1)(n + 2)(2n + 3)}{6}$$

Careful with factoring please!!!

- By leveraging associativity of sum, the LHS can be written as follows:

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^n i^2 + (n + 1)^2$$

Inductive Step

- We will now attempt to prove $P(n + 1)$, i.e

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- By leveraging associativity of sum, the LHS can be written as follows:

$$\sum_{i=1}^{n+1} i^2 = \left(\sum_{i=1}^n i^2 \right) + (n + 1)^2$$

We can apply the IH here!

Inductive Step

- By IH, we can now write:

$$\sum_{i=1}^{n+1} i^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

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- By IH, we can now write:

$$\sum_{i=1}^{n+1} i^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

- Remember: we **want** this to be equal to

$$\frac{(n+1)(n+2)(2n+3)}{6}$$

- We will fearlessly manipulate the algebra until it does!

Inductive Step - Algebra

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6} \\ &= \frac{(n+1)[n(2n+1) + 6(n+1)]}{6} = \frac{(n+1)[2n^2 + 7n + 6]}{6} \end{aligned}$$

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- If only we could prove that $2n^2 + 7n + 6 = (n+2)(2n+3)$, we'd be done!

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- If only we could prove that $2n^2 + 7n + 6 = (n+2)(2n+3)$, we'd be done!
- But.... $(n+2)(2n+3) = 2n^2 + 3n + 4n + 6 = 2n^2 + 7n + 6!$
- **So we're done.**

Sums of Powers of 2

- Prove that the sum of the first n terms of a **geometric sequence** with $a_1 = 1$ is equal to $2^n - 1$.

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- Prove that the sum of the first n terms of a **geometric sequence** with $a_1 = 1$ is equal to $2^n - 1$.
- Symbolically:

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

Proof

- Proof : We attempt to prove $P(n)$, $\forall n \in \mathbb{N}$. We proceed via **induction on n** .

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- **Inductive base:** We attempt to prove $P(1)$.

$$P(1): \sum_{i=0}^{1-1} 2^i = 2^1 - 1 \Leftrightarrow 1 = 1$$

So $P(1)$ is true.

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So $P(1)$ is true.

- **Inductive hypothesis:** Suppose $n \geq 0$. We assume $P(n)$, i.e

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

Proof (contd.)

- **Inductive step:** We will attempt to prove $P(n + 1)$, i.e

$$\sum_{i=0}^{(n+1)-1} 2^i = 2^{n+1} - 1$$

From the LHS to the RHS:

$$LHS = \sum_{i=0}^n 2^i = \sum_{i=0}^{n-1} 2^i + 2^n = 2^n - 1 + 2^n = 2(2^n) - 1 = 2^{n+1} - 1 = RHS \quad \square$$

Sums of Powers of m

- Prove that the sum of the first n terms of a **geometric sequence** with $m \in (\mathbb{R} - \{1\})$ and $a_1 = 1$ is equal to $\frac{m^n - 1}{m - 1}$.

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Note: In the base case we are assuming $m \neq 1$

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- **Inductive hypothesis:** Suppose $n \geq 0$. We assume $P(n)$, i.e

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Proof (contd.)

- **Inductive step:** We will attempt to prove $P(n + 1)$, i.e

$$\sum_{i=0}^{(n+1)-1} m^i = \frac{m^{n+1} - 1}{m - 1}$$

From the LHS to the RHS:

$$\begin{aligned} \text{LHS} &= \sum_{i=0}^n m^i = \sum_{i=0}^{n-1} m^i + m^n = \frac{m^n - 1}{m - 1} + m^n = \frac{m - 1 + m^n(m - 1)}{m - 1} = \frac{m^{n+1} - 1}{m - 1} = \text{RHS} \quad \square \end{aligned}$$

By the IH

Base Cases

- It is standard to change your base cases to later in your index if the theorem you are trying to prove starts later

COIN PROBLEMS!

A Coin Problem

- We will prove that every dollar amount ≥ 4 cents can be **exclusively** paid for by 2 and/or 5 cent coins.

Theorem Expressed in Quantifiers

- All quantifiers implicitly assumed over \mathbb{N} .

$$(\forall n \geq 4)(\exists n_1, n_2)[n = 2n_1 + 5n_2]$$

Inductive Base

- The least amount of money we are required to prove the statement for is 4¢, so we will attempt to **prove $P(4)$** .

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- The least amount of money we are required to prove the statement for is 4¢, so we will attempt to **prove $P(4)$** .
- For $n = 4$, we have 4¢. Since $4¢ = 2 \times 2¢$, we are done (we have shown that the amount of 4¢ can be **exclusively** paid for by using only 2 **and/or** 5 cent coins)

Inductive Hypothesis

- Let $n \geq 4$.
- Assume $P(n) \Leftrightarrow (\exists n_1, n_2)[n = 2n_1 + 5n_2]$

Inductive Step

- We will prove that $P(n) \Rightarrow P(n + 1)$, i.e that we can pay an amount of money equal to $n + 1$ cents using **only 2¢ or 5¢ coins**.

Inductive Step

- We will prove that $P(n) \Rightarrow P(n + 1)$, i.e that we can pay an amount of money equal to $n + 1$ cents using **only 2¢ or 5¢ coins**.
- In terms of algebra, what we want to prove is:

$$(\exists n_3, n_4 \in \mathbb{N}) [n + 1 = 2n_3 + 5n_4]$$

Different variables from IH!

Inductive Step (contd.)

- From the **Inductive Hypothesis (IH)**, we have that for some specific positive integers n_1 and n_2 :

$$n = 2n_1 + 5n_2$$

Inductive Step (contd.)

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1. Case #1: $n_1 \geq 2$

- I have at least two 2¢ coins, so I can take away two 2¢ coins and add one 5¢ coin

Inductive Step (contd.)

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1. Case #1: $n_1 \geq 2$

- I have at least two 2¢ coins, so I can take away two 2¢ coins and add one 5 ¢ coin
- By adding 1 on both sides of the IH we obtain:

$$\begin{aligned} n + 1 &= 2n_1 + 5n_2 + 1 = 2n_1 + 5n_2 + (5 - 2 * 2) = \\ &= (2n_1 - 4) + (5n_2 + 5) = 2 \underbrace{(n_1 - 2)}_{n_3} + 5 \underbrace{(n_2 + 1)}_{n_4} = 2n_3 + 5n_4 \end{aligned}$$

Inductive Step (contd.)

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- I have at least two 2¢ coins, so I can take away two 2¢ coins and add one 5 ¢ coin
- By adding 1 on both sides of the IH we obtain:

$$\begin{aligned} n + 1 &= 2n_1 + 5n_2 + 1 = 2n_1 + 5n_2 + (5 - 2 * 2) = \\ &= (2n_1 - 4) + (5n_2 + 5) = 2 \underbrace{(n_1 - 2)}_{n_1 - 2 \geq 0 \text{ because } n_1 \geq 2} + 5 \underbrace{(n_2 + 1)}_{\text{In } \mathbb{N} \text{ by closure}} = 2n_3 + 5n_4 \end{aligned}$$

$n_1 - 2 \geq 0$ because
 $n_1 \geq 2$

In \mathbb{N} by closure

Inductive Step

2. Case #2: $n_2 \geq 1$

- I have at least one 5¢ coin so I can take away one 5¢ coin and add three 2¢ coins

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$$\begin{aligned}n + 1 &= 2n_1 + 5n_2 + 1 = 2n_1 + 5n_2 + (3 * 2 - 5) = \\ &= 2 \underbrace{(n_1 + 3)}_{n_3} + 5 \underbrace{(n_2 - 1)}_{n_4} = 2n_3 + 5n_4\end{aligned}$$

Inductive Step

2. Case #2: $n_2 \geq 1$

- I have at least one 5¢ coin so I can take away one 5¢ coin and add three 2¢ coins
- By adding 1 on both sides of the IH we obtain:

$$\begin{aligned} k + 1 &= 2k_1 + 5k_2 + 1 = 2k_1 + 5k_2 + (3 * 2 - 5) = \\ &= \underbrace{2(n_1 + 3)} + 5\underbrace{(n_2 - 1)} = 2n_3 + 5n_4 \end{aligned}$$

$(n_1 + 3) \in \mathbb{N}$
by closure

$n_2 - 1 \geq 0$
because
 $n_2 \geq 1$

Inductive Step

3. Case #3: $(n_1 \leq 1) \wedge (n_2 = 0)$

- This case means that we have either 0 or 2¢ at our disposal.

Inductive Step

3. Case #3: $(n_1 \leq 1) \wedge (n_2 = 0)$

- This case means that we have either 0 or 2¢ at our disposal.
- But this is not possible, since we want to prove the theorem only for values $\geq 4\text{¢}$
- So we're done. \square

A Coin Problem for You!

Prove to me that every dollar amount ≥ 20 cents can be **exclusively** paid for through combinations of **5**-cent coins and **6**-cent coins!

Go to Breakout Rooms

TREATING INEQUALITIES

What if your theorem only holds when $n \geq 4$?

- We want to compare 2^n and $n!$.

n	2^n	$n!$
1	2	1
2	4	2
3	8	6
4	16	24

What if your theorem only holds when $n \geq 4$?

- We want to compare 2^n and $n!$.

n	2^n	$n!$
1	2	1
2	4	2
3	8	6
4	16	24

- It seems like $(\forall n \geq 4)[2^n < n!]$
- Our current Induction Principle cannot handle this!
- Don't Panic!

Modified Induction Principle

- From
 - Base Case (BC): $P(a)$
 - Induction Step (IS): $\forall n \geq a, P(n) \implies P(n + 1)$

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Modified Induction Principle

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 - Base Case (BC): $P(a)$
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- We can deduce
 - $\forall n \geq a, P(n)$
- Why does the Modified Induction Principle Work?
 - Similar to who the original Induction Principle worked.

Here's One with an Inequality!

- Prove that for all integers n at least 4, $2^n < n!$
- 1. **IB:** We will prove $P(4) \Leftrightarrow 2^4 < 4!$ Done.

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 1. **IB:** We will prove $P(4) \Leftrightarrow 2^4 < 4!$ Done.
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Here's One with an Inequality!

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 1. **IB:** We will prove $P(4) \Leftrightarrow 2^4 < 4!$ Done.
 2. **IH:** For $n \geq 4$, we assume $P(n)$, i.e $2^n < n!$
 3. **IS:** We will prove $P(n) \Rightarrow P(n + 1)$, i.e

$$(2^n < n!) \Rightarrow (2^{n+1} < (n + 1)!)$$

Inductive Step...

- Prove that for all integers n at least 4, $2^n < n!$
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 2. **IH:** For $n \geq 4$, we assume $P(n)$, i.e $2^n < n!$
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 - Since $n \geq 4$, we have that $2 < n + 1 \stackrel{n!>0}{\Leftrightarrow} n! \cdot 2 < n! (n + 1)$ (3)
 - $(2) \stackrel{(3)}{\Rightarrow} 2^n \cdot 2 < (n + 1)! \stackrel{(1)}{\Leftrightarrow} 2^{n+1} < (n + 1)!$

STOP

RECORDING