Predicate and Quantifier Review

250H

Negating Quantified Expressions

Negation	Equivalent Statement	When Is Negation True?	When False?
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, P (x) is false	There is an x for which P (x) is true
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which P (x) is false	P (x) is true for every x

The Order of Quantifiers

- Order Matters
  - Unless all quantifiers are universal quantifiers or all are existential quantifiers
- The statements $\exists y \forall x P(x, y)$ and $\forall x \exists y P(x, y)$ are not logically equivalent
  - The statement $\exists y \forall x P(x, y)$ is true if and only if there is a y that makes P (x, y) true for every x.
  - There must be a particular value of y for which P (x, y) is true regardless of the choice of x.
  - $\forall x \exists y P(x, y)$ is true if and only if for every value of x there is a value of y for which P (x, y) is true
  - No matter which x you choose, there must be a value of y (possibly depending on the x you choose) for which P (x, y) is true
    - $\forall x \exists y P(x, y)$: y can depend on x
    - $\exists y \forall x P(x, y)$: y is a constant independent of x

Logical Operator: Conditional Statements

Common ways to express $p \rightarrow q$:

- if p, then q
- p implies q
- if p, q
- p only if q
- p is sufficient for q
- a sufficient condition for q is p
- q if p
- q whenever p
- q when p
- q is necessary for p
- a necessary condition for p is q
- q follows from p
- q unless $\neg p$

Example 1: Translating Math Statements into Statements

- Translate the statement “The sum of two positive integers is always positive” into a logical expression
  - Rewrite it so that the implied quantifiers and a domain are shown
    - For every two integers, if these integers are both positive, then the sum of these integers is positive.
  - Introduce the variables x and y to obtain
    - For all positive integers x and y, x + y is positive.
  - Quantify
    - $\forall x \forall y((x > 0) \land (y > 0) \rightarrow (x + y > 0))$, where the domain for both variables consists of all integers.
    - Alternate Solution: $\forall x \forall y(x + y > 0)$, where the domain for both variables consists of all positive integers.

Example 2: Translating Math Statements into Statements

- Translate the statement: Every real number except zero has a multiplicative inverse. *(A multiplicative inverse of a real number x is a real number y such that xy = 1.)*
    - Rewrite it so that the implied quantifiers and a domain are shown
        - For every real number x except zero, x has a multiplicative inverse.
    - Introduce the variables x and y to obtain
        - For every real number x, if $x \neq 0$, then there exists a real number y such that $xy = 1$
    - Quantify
        - $\forall x((x \neq 0) \rightarrow \exists y(xy = 1))$

Example 3: Translating Math Statements into Statements

- Translate the statement: There exists two distinct rational numbers such that xy = 0.
  - $\exists x, y \in \mathbf{Q}\ ((x \neq y) \land (xy = 0))$

Example 4: Translating Math Statements into Statements

- Translate the statement: There exists an infinite number of natural numbers.
  - $\forall x \in \mathbf{N}\ \exists y \in \mathbf{N}\ (y > x)$

Example 5: Translating Math Statements into Statements

- Translate the statement: There are no natural numbers x, y such that xy = -1.
    - $\neg(\exists x,y\ (xy = -1))$
    - $\forall x, y \in \mathbf{N}\ (xy \neq -1)$