BILL AND EMILY RECORD LECTURE!!!!

Problems with a Point: Exploring Math and Computer Science

Authors:
William Gasarch
Clyde Kruskal

How This Book Came to Be

Book's Origin

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To quote Ralph Waldo Emerson

A foolish consistency is the hobgoblin of small minds.

Possible Subtitles

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The publisher wisely decided to be less cute and more informative:
Problems with a Point: Exploring Math and Computer Science

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Please Procure People to Polish Prose and Proofs of Problems with a Point

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Now onto some samples of the book!

Point: Students Can Give Strange Answers

The Paint Can Problem

From the Year 2000 Maryland Math Competition:
There are 2000 cans of paint. Show that at least one of the following two statements is true:

- There are at least 45 cans of the same color.
- There are at least 45 cans that are different colors.

Work on it in groups! Prove a General Theorem.

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Answer:
If there are 45 different colors of paint then we are done. Assume there are ≤ 44 different colors. If all colors appear ≤ 44 times then there are $44 \times 44 = 1936 < 2000$ cans of paint, a contradiction.

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Note: this was Problem 1, which is supposed to be easy and indeed 95% got it right. What about the other 5%? Next slide.

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ANSWER:

Paint cans are grey. Hence there are all the same color. Therefore there are 2000 cans that are the same color.

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If you look at a paint color really really carefully there will be differences. Hence, even if two cans seem to both be (say) RED, they are really different. Therefore there are 2000 cans of different colors.

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A Triangle Problem

From the year 2007 Maryland Math Competition.

QUESTION *Let ABC be a fixed triangle. Let COL be any 2-coloring of the plane where each point is colored with red or green. Prove that there is a triangle DEF in the plane such that DEF is similar to ABC and the vertices of DEF all have the same color.*

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Note I think I was assigned to grade it since it **looks like** the kind of problem I would make up, even though I didn't. It was problem 5 (out of 5) and was hard. About 100 students tried it, 8 got full credit, 10 got partial credit

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All the vertices are red because I can make them whatever color I want. I can also write at a 30 degree angle to the bottom of this paper (The students answer was written at a 30 degree angle to the bottom of the paper.) if thats what I feel like doing at the moment. Just like $2 + 2 = 5$ *if thats what my math teacher says. Math is pretty subjective anyway.*

Was Student One Serious?

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Theorem The students is not serious.

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Theorem The students is not serious.
Proof Assume, by contradiction, that they are serious. Then they really think math is subjective. Hence they don't really understand math. Hence they would not have done well enough on Part I to qualify for Part II. But they took Part II. Contradiction.

Funny Answers Two

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I like to think that we live in a world where points are not judged by their color, but by the content of their character. Color should be irrelevant in the the plane. To prove that there exists a group of points where only one color is acceptable is a reprehensible act of bigotry and discrimination.

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Was Student Two Serious? Yes. About **Justice!**.

The Real Answer to Points in the Plane Problem

Each point in the plane is colored either red or green. Let ABC be a fixed triangle. Prove that there is a triangle DEF in the plane such that DEF is similar to ABC and the vertices of DEF all have the same color.

Fix a 2-coloring of the plane.

There are 3 equally-spaced mono points on x-axis

Proof Clearly there are two points on the x-axis of the same color: p_1, p_2 are RED. If p_3, the midpoint of p_1, p_2, is RED then p_1, p_3, p_2 are all RED. DONE. Hence we assume p_3 is GREEN.

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Let p_4 be such that $|p_1 - p_4| = |p_2 - p_1|$. If p_4 is RED then p_4, p_1, p_2 are all RED. DONE. Hence we assume p_4 is GREEN.

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Only case left p_3, p_4, p_5 are all GREEN. DONE.

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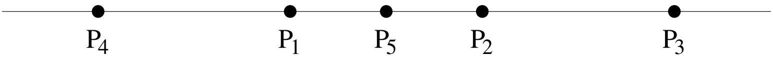

Finish Proof By Picture

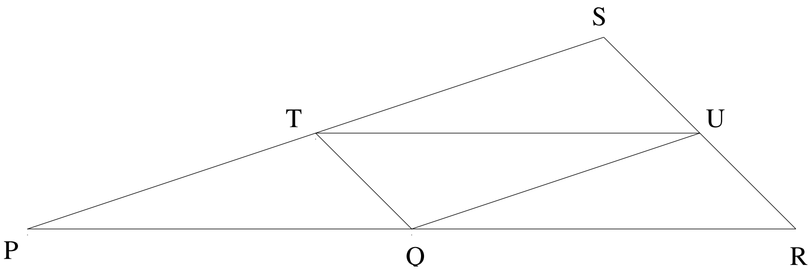

Figure: Triangle Similar to ABC with Monochromatic Vertices

P, Q, R are RED.

If T or U or S are RED then get RED Triangle similar to ABC.

If not then ALL of T, U, S are GREEN, so get GREEN triangle similar to ABC.

Point: What is a Pattern?

Simple Functions

Bill assigned the following in Discrete Math: For each of the following sequences find a **simple function** $A(n)$ such that the sequence is $A(1), A(2), A(3), \ldots$

1. 10, -17, 24, -31, 38, -45, 52, $\cdots$
2. -1, 1, 5, 13, 29, 61, 125, $\cdots$
3. 6, 9, 14, 21, 30, 41, 54, $\cdots$

Caveat: These are NOT trick questions.
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1. 10, -17, 24, -31, 38, -45, 52, $\cdots$ $A(n) = (-1)^{n+1}(7n+3)$.
2. -1, 1, 5, 13, 29, 61, 125, $\cdots$ $A(n) = 2^n - 3$.
3. 6, 9, 14, 21, 30, 41, 54, $\cdots$ $A(n) = n^2 + 5$.

A Student asks — What is a Simple Function?

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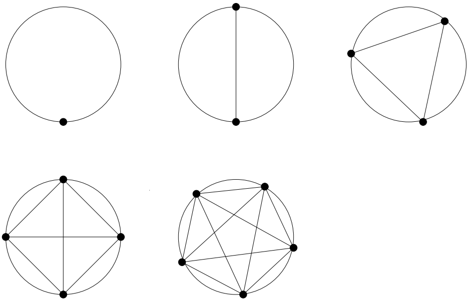
The student got the first one right, but left the other two blank.

When Do Patterns Hold?

The last question brings up the question of when patterns do and don't hold. We looked for cases where a pattern *did not* hold.

First Non-Pattern: n Points on a circle

What is the max number of regions formed by connecting every pair of n points on a circle. For $n = 1, 2, 3, 4, 5$:



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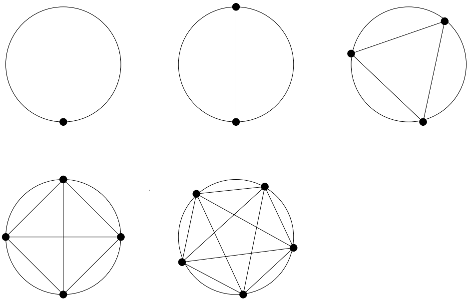

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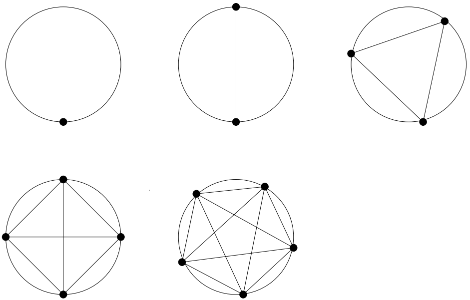

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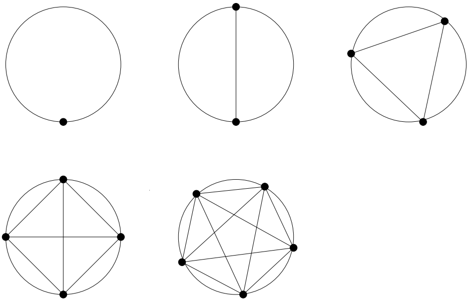

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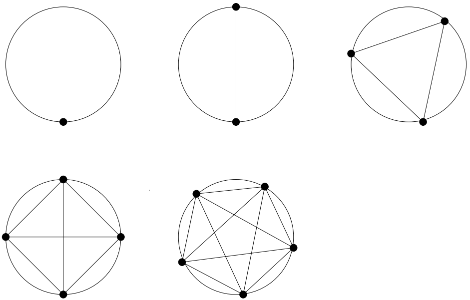

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But for $n = 6$, the number of regions is only 31.
The actual number of regions for n points is $\binom{n}{4} + \binom{n}{2} + 1$.

Second Non-Pattern: Borwein Integrals

$$\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin x}{x} \frac{\sin \frac{x}{3}}{\frac{x}{3}} = \frac{\pi}{2}$$

$$\vdots$$

$$\int_0^\infty \frac{\sin x}{x} \frac{\sin \frac{x}{3}}{\frac{x}{3}} \frac{\sin \frac{x}{5}}{\frac{x}{5}} \frac{\sin \frac{x}{7}}{\frac{x}{7}} \frac{\sin \frac{x}{9}}{\frac{x}{9}} \frac{\sin \frac{x}{11}}{\frac{x}{11}} \frac{\sin \frac{x}{13}}{\frac{x}{13}} = \frac{\pi}{2}$$

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$$\int_0^\infty \frac{\sin x}{x} \frac{\sin \frac{x}{3}}{\frac{x}{3}} = \frac{\pi}{2}$$

$$\vdots$$

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But

$$\int_0^\infty \frac{\sin x}{x} \frac{\sin \frac{x}{3}}{\frac{x}{3}} \frac{\sin \frac{x}{5}}{\frac{x}{5}} \frac{\sin \frac{x}{7}}{\frac{x}{7}} \frac{\sin \frac{x}{9}}{\frac{x}{9}} \frac{\sin \frac{x}{11}}{\frac{x}{11}} \frac{\sin \frac{x}{13}}{\frac{x}{13}} \frac{\sin \frac{x}{15}}{\frac{x}{15}} =$$

$$\frac{467807924713440738696537864469\pi}{935615849440640907310521750000} \sim 0.99999999998529371 86 \times \frac{\pi}{2}$$

Why the breakdown at 15?

Because

$$\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{13} < 1$$

but

$$\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{15} > 1.$$

Another Non-Pattern: More Borwein Integrals

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$$\int_0^\infty 2\cos(x)\frac{\sin x}{x}\frac{\sin\frac{x}{3}}{\frac{x}{3}}\cdots\frac{\sin\frac{x}{113}}{\frac{x}{113}} < \frac{\pi}{2}$$

Why the breakdown at 113?

Because

$$\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{111} < 2$$

but

$$\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{113} > 2.$$

Computers to FIND proofs vs Computers to DO Proofs

Colorings and Square Differences

The following are all true:

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The proofs in the literature of these theorems give EEEEEEEEEENORMOUS bounds on W_2, W_3, W_4, W_c. We look at easier proofs with two **points** in mind:

- Would they be good questions on a HS math competition?
- What is the role of Computers in these proofs?

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AH-HA: $\mathrm{COL}(1) = \mathrm{COL}(5)$ and $5 - 1 = 4 = 2^2$. So $W_2 \leq 5$.

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So $W_2 = 4$.

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Upshot Could be easy HS Math Comp Prob. No computer used.

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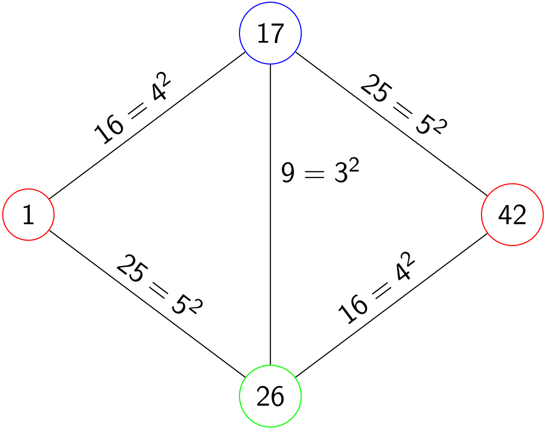

Figure: $\mathrm{COL}(x) = \mathrm{COL}(x + 41)$

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Can we get better bound on W_3?

Better Bound on W_3

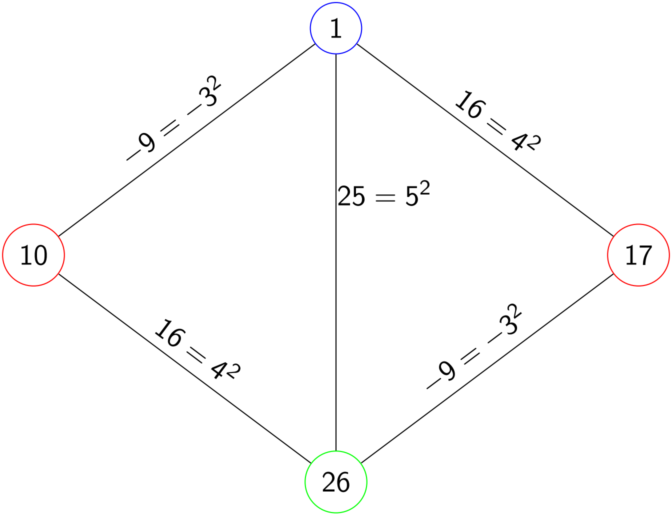

Figure: If $x \geq 10$ then $\mathrm{COL}(x) = \mathrm{COL}(x+7)$, so $W_3 \leq 59$

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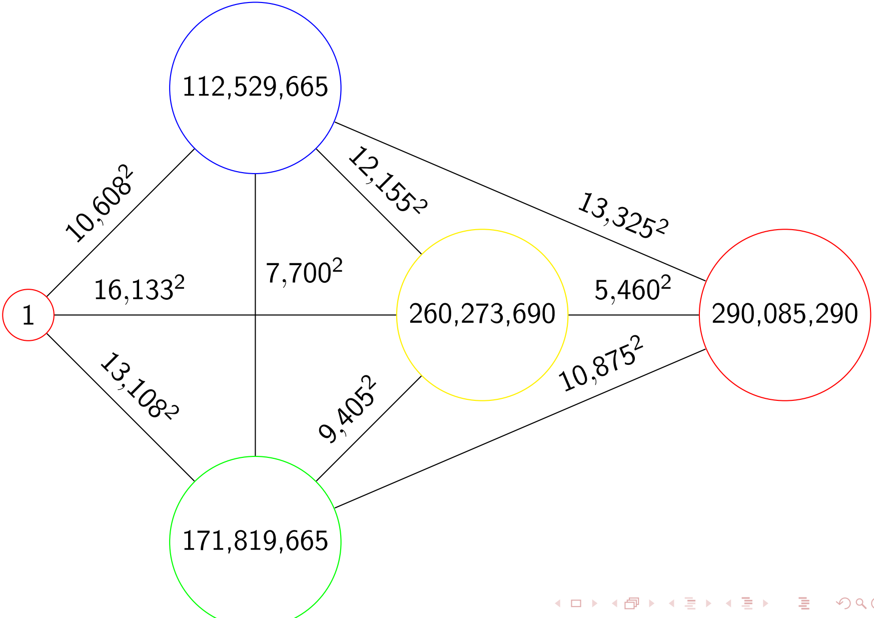
W_4 Exists: $\mathrm{COL}(x) = \mathrm{COL}(x + 290{,}085{,}290)$



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3. A Computer Search showed that $W_4 = 58$.
   PRO Get exact value.
   CON not human-verifiable. Does not generalize to W_5.

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