START

RECORDING

Discrete Probability Part 2

CMSC 250

Dependent and Independent Events

Independent Events *(informally)*

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- Examples:
  - The event E1 = “first coin toss” and E2 = “second coin toss”
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  - Jason flips a coin and then picks a card.

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    - The event E1 = “first coin toss” and E2 = “second coin toss”
    - With the same die, the events E1 = “roll 1”, E2 = “roll 2”, E3 = “roll 3”
    - Jason flips a coin and then picks a card.
- Counter-examples:
    - E1 = “Die is even”, E2=“Die is 6”
    - E1= “Grade in 250” and “Passing 250”

Law of Joint Probability *(informally)*

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- Formally, we define that A and B are independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

Disjoint or Independent?

1. $E_1 =$ “It rains in College Park, MD today”
   $E_2 =$ “It rains in Athens, Greece today”

Disjoint

Independent

Both

Neither

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   E_2 = Student passes the class

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Recap: “Disjoint” vs “Independent”

- Friends don’t let friends get confused between “disjoint” and “independent”!

Disjoint	**Independent**
Has a set-theoretic interpretation!	Has a causality interpretation!
Means that $P(A \cap B) = 0$	Means that $P(A \cap B) = P(A) \cdot P(B)$
Means that $P(A \cup B) = P(A) + P(B)$	Means that $P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$

Disjoint Probability (“OR” of Two Events)

- Jason rolls two dice.
  - **What is the probability that he rolls a 7 or a 9?**

Disjoint Probability (“OR” of Two Events)

- Jason rolls two dice.
  - **What is the probability that he rolls a 7 or a 9?**
  - #Ways to roll a 7 is 6.
  - #Ways to roll a 9 is 4: (6, 3), (5, 4), (4, 5), (3, 6)
  - #Ways to roll a 7 OR a 9 is then 10.
  - Therefore, the probability is $\frac{10}{36} = \frac{5}{18}$
  - Key: Rolling a 7 and a 9 are **disjoint events**.

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- How big is $Face\_Card \cup Hearts$?

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- How big is $Face\_Card \cup Hearts$ (abbrv. F, H below)?
  - Use law of inclusion / exclusion!

$$|F \cup H| = |F| + |H| - |F \cap H| = 12 + 13 \; - \mathbf{3} = \mathbf{22}$$

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- 52-card deck
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$$|F \cup H| = |F| + |H| - |F \cap H| = 12 + 13\ - \mathbf{3} = \mathbf{22}$$

- So probability = $\frac{22}{52} = \frac{11}{26}$.

Alternative Viewpoint

- $P(F) = \frac{12}{52}$
- $P(H) = \frac{13}{52}$
- $P(F \cap H) = \frac{3}{52}$
- $P(F \cup H) = P(F) + P(H) - P(F \cap H)$

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- We can also do:

$$\frac{\binom{13}{4}\binom{4}{1}\binom{4}{2} * 4^3}{\binom{52}{5}}$$

Probability of Unions

$$P(A \cup B) = P(A) + P(B) - P\ (A \cap B)$$

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- If A and B are disjoint, we have

$$P(A \cup B) = P(A) + P(B)$$

Probability of Unions of 3 Sets

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$
$$- \quad P\,(A \cap B) - P(B \cap C) - P(A \cap C)$$
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- If A, B and C are pairwise independent , we have :

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A) \cdot P(B) - P(B) \cdot P(C) -$$
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- If A, B and C are pairwise disjoint (so $A \cap B = A \cap C = B \cap C = \emptyset$, so clearly $A \cap B \cap C = \emptyset$), we have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

Conditional Probability and Bayes’ Law

Conditional Probability

- If A occurs, then is B
  a) More likely?
  b) Equally likely?
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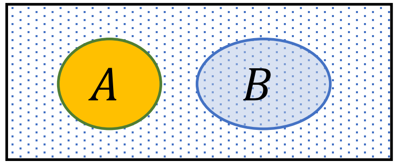
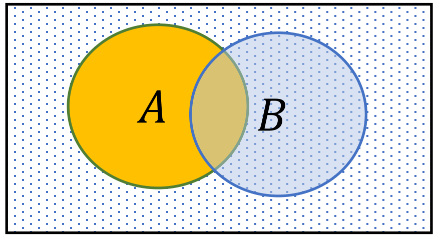


OR



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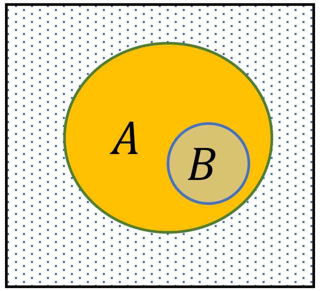

OR …

Examples

- We roll two dice
  - Event A = “Sum of the dice $S \equiv 0 \ (mod\ 4)$”
    - Note that $P(A) = \frac{9}{36} = \frac{1}{4}$, since we have nine rolls of the dice that sum to a multiple of 4:

      (1, 3), (2, 2), (3, 1), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (6, 6)

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  - Outcomes of B are (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)
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- However, once B occurs, instead of 36 outcomes, we now have… 6 outcomes.
  - Only 2 of them are outcomes that correspond to A.
  - Therefore, the probability of A given B is $\frac{2}{6} = \frac{1}{3}$

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    - Event A = “Sum of the dice is $\geq$ 8”
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Examples

- We once again two roll dice
  - Event A = “Sum of the dice is ≥ 8”
  - Event B = “ First die is 4”
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Go up

Go down

Stay the same

Unknown to science

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Unknown to science

Let’s see if your intuition was correct!

Examples

- We once again two roll dice
  - Event A = “Sum of the dice is $\geq$ 8” $P(A) = ?$ (work on it)
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Examples

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Examples

- We once again two roll dice
  - Event A = "Sum of the dice is ≥ 8" $P(A) = \frac{15}{36} = \frac{5}{12}$
  - Event B = "First die is a 4" $P(B) = \frac{1}{6}$
- Prob of A given B = Prob second dice is 4, 5, or 6 = $\frac{3}{6} = \frac{1}{2} > \frac{5}{12}$

By just $\frac{1}{12}$…

[✓ Go up] Go down Stay the same Unknown to science

Conditional Probability

- Let A, B be two events. The conditional probability of A ***given*** B, denoted $P(A \mid B)$ is defined as follows:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Re-Thinking Independent Events

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- Applying the definition of $P(A|B)$ we have:
  - $\frac{P(A \cap B)}{P(B)} = P(A) \Rightarrow P(A \cap B) = P(A) \cdot P(B)$, which is a relationship we had reached **earlier** when discussing the joint probability.

Complex Probabilities

- Suppose that I have two dice: a six-sided one and a ten-sided one.

Complex Probabilities

- Suppose that I have two dice: a six-sided one and a ten-sided one.
- I pick either one of them with probability $\frac{1}{2}$ and roll it.
  - What's the probability that the die comes up 6? (work on this yourselves **NOW**)

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$$P(Roll = 6) = P(Roll = 6, Die = 6) + P(Roll = 6, Die = 10) =$$

$$= P(Roll = 6 | Die = 6) \times P(Die = 6) + P(Roll = 6 | Die = 10) \times P(Die = 10) =$$

$$= \frac{1}{6} \times \frac{1}{2} + \frac{1}{10} \times \frac{1}{2} = \frac{1}{12} + \frac{1}{20} = \frac{2}{15} \approx 0.1333\ldots$$

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- What’s the probability that I come up with a 6?

$$P(Roll = 6) = P(Roll = 6, Die = 6) + P(Roll = 6, Die = 10) =$$

$$= P(Roll = 6 | Die = 6) \times P(Die = 6) + P(Roll = 6, Die = 10) \times P(Die = 10) =$$

$$= \frac{1}{6} \times \frac{4}{9} + \frac{1}{10} \times \frac{5}{9} = \frac{2}{27} + \frac{1}{18} = \frac{7}{54} \approx 0.130 < \mathbf{0.133}$$

Bayes' Law

- Suppose A and B are events in a sample space Ω. Then, the following is an identity:

$$P(A|B) = P(B|A)\frac{P(A)}{P(B)}$$

known as **Bayes' Law**

Questions

- If $P(A|B) = P(A)$, is it the case that $P(B|A) = P(B)$?

Yes

No

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[✓ Yes] No

- Substituting $P(A|B)$ with $P(A)$ in the formulation of Bayes' Law, we have:

$$\cancel{P(A)} = P(B \mid A) \cdot \frac{\cancel{P(A)}}{P(B)} \Rightarrow 1 = \frac{P(B|A)}{P(B)} \Rightarrow P(B|A) = P(B)$$

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[✓ Yes] No

(A *ind* B) iff (B *ind* A)

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Questions

- If $P(B) = 0$, then is $P(A|B)$ also 0?

Yes

No

Questions

- If $P(B) = 0$, then is $P(A|B)$ also 0?

Yes [✓ No]

- It is **undefined**, since $P(A \mid B) = P(B \mid A) \cdot \frac{P(A)}{P(B)}$

STOP RECORDING