The Birthday Paradox

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What is Your Birthday?

The Birthday Paradox

- What is the minimum number of people who need to be in a room so that the probability that at least two of them have the same birthday is greater than 1/2?
  - We assume that the birthdays of the people in the room are independent
  - We assume that each birthday is equally likely and that there are 366 days in the year
  - To find the probability that at least two of n people in a room have the same birthday, we first calculate the probability p_n that these people all have different birthdays
  - The probability that at least two people have the same birthday is $1 - p_n$
  - To find p_n, consider the birthdays of the n people in some fixed order
  - Imagine them entering the room one at a time
  - We will compute the probability that each successive person entering the room has a birthday different from those of the people already in the room

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- The birthday of the first person certainly does not match the birthday of someone already in the room
- The probability that the birthday of the second person is different from that of the first person is 365/366 because the second person has a different birthday when he or she was born on one of the 365 days of the year other than the day the first person was born.
- The probability that the third person has a birthday different from both the birthdays of the first and second people given that these two people have different birthdays is 364/366
- In general, the probability that the jth person, with $2 \leq j \leq 366$, has a birthday different from the birthdays of the $j - 1$ people already in the room given that these $j - 1$ people have different birthdays is

$$\frac{366-(j-1)}{366} = \frac{367-j}{366}$$

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- Because we have assumed that the birthdays of the people in the room are independent, we can conclude that the probability that the n people in the room have different birthdays is

$$p_n = \frac{365}{366}\frac{364}{366}\cdots\frac{367-n}{366}$$

- It follows that the probability that among n people there are at least two people with the same birthday is

$$1 - p_n = 1 - \left(\frac{365}{366}\frac{364}{366}\cdots\frac{367-n}{366}\right)$$

- To determine the minimum number of people in the room so that the probability that at least two of them have the same birthday is greater than 1/2, we use the formula we have found for $1 - p_n$ to compute it for increasing values of n until it becomes greater than 1/2

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- After considerable computation we find that for $n = 22$, $1 - p_n \approx 0.475$, while for $n = 23$, $1 - p_n \approx 0.506$
- The minimum number of people needed so that the probability that at least two people have the same birthday is greater than 1/2 is 23