Homework 09 Solutions

Problem 2: Binomial Identity

Claim

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Solution

LHS: This is the number of ways to choose k objects from n objects.
RHS: Consider a set of n objects, from which we are selecting k. Fix one distinguished element x.

- *Case 1: x is included $\Rightarrow$ choose remaining $k-1$ from $n-1$: $\binom{n-1}{k-1}$*
- *Case 2: x is not included $\Rightarrow$ choose all k from $n-1$: $\binom{n-1}{k}$*

Adding both cases gives the identity.

Problem 3(a): Three 2-of-a-Kinds

Solution

Choose 3 distinct ranks:

$$\binom{13}{3}$$

Choose 2 suits for each rank:

$$\binom{4}{2}^3$$

Total hands:

$$\binom{52}{6}$$

Probability:

$$\frac{\binom{13}{3}\binom{4}{2}^3}{\binom{52}{6}}$$

Problem 3(b): 4-of-a-Kind + 2-of-a-Kind

Solution

Choose rank for 4-of-a-kind:

$$\binom{13}{1}$$

Choose rank for pair:

$$\binom{12}{1}$$

Choose suits for pair:

$$\binom{4}{2}$$

Probability:

$$\frac{\binom{13}{1}\binom{12}{1}\binom{4}{2}}{\binom{52}{6}}$$

Problem 3(c): Two 3-of-a-Kinds

Solution

Choose 2 ranks:

$$\binom{13}{2}$$

Choose 3 suits for each:

$$\binom{4}{3}^2$$

Probability:

$$\frac{\binom{13}{2}\binom{4}{3}^2}{\binom{52}{6}}$$

Problem 4: No Fair Loaded 8-Sided Dice

Proof.

Let the loaded dice have probability generating functions

$$P(x) = p_1 + p_2 x + p_3 x^2 + \cdots + p_8 x^7, \qquad Q(x) = q_1 + q_2 x + q_3 x^2 + \cdots + q_8 x^7,$$

where $p_i, q_i \geq 0$ and

$$P(1) = Q(1) = 1.$$

The coefficient of x^j in $P(x)Q(x)$ is the probability that the two dice sum to $j + 2$. Thus, if all sums $2, 3, \ldots, 16$ were equally likely, then

$$P(x)Q(x) = \frac{1}{15}(1 + x + x^2 + \cdots + x^{14}).$$

Now

$$1 + x + x^2 + \cdots + x^{14} = \frac{x^{15} - 1}{x - 1}.$$

$\square$

Problem 4 Cont.

Proof.

Therefore the roots of

$$1 + x + x^2 + \cdots + x^{14}$$

are the nontrivial 15th roots of unity. These are complex roots.
Thus $P(x)Q(x)$ has complex roots, so at least one of $P(x)$ or $Q(x)$ must have a complex root.
But $P(x)$ and $Q(x)$ are probability generating functions for loaded dice, so their coefficients are nonnegative real numbers and sum to 1. This contradicts the required factorization into the complex roots of

$$1 + x + x^2 + \cdots + x^{14}.$$

Therefore, there is no way to load two 8-sided dice so that all sums $2, 3, \ldots, 16$ are equally likely. $\square$

Problem 5(a): Standard Dice Probabilities

Number of outcomes: 64.
For sum i:

$$P(i) = \frac{\#\text{ways to get } i}{64}$$

Triangular distribution:

- Increasing for $i = 2$ to 9
- Decreasing for $i = 9$ to 16

Problem 5(b): Nonstandard Dice

Using generating functions:

$$(x + x^2 + \cdots + x^8)^2$$

Factorization yields valid dice:
Example:
Dice 1: $(5, 4, 4, 3, 3, 2, 2, 1)$
Dice 2: $(11, 9, 7, 7, 5, 5, 2, 1)$
These produce identical sum probabilities.