Homework 06, MORALLY Due March 24

1. (0 points) What is your name.

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2. (25 points) In this problem we will guide you through the proof that the following set is infinite

$$X = \{p\colon p \equiv 3 \pmod 4 \text{ and } p \text{ is prime }\}.$$

We begin the proof for you but you will need to finish it.

Assume X is finite. So let $X = \{p_1, \ldots, p_k\}$.

Let

$N = 4p_1p_2 \cdots p_k - 1$

If N is prime then you have found a prime NOT on the list. You are done!

If N is not prime then let $N = q_1^{a_1} \cdots q_m^{a_m}$. where $q_1, \ldots, q_m$ are primes.

YOU NEED TO SHOW THAT at least one of the q_i is $\equiv 3 \pmod 4$.

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3. (25 points) Let a_n be defined as follows

$a_0 = 1$

$a_1 = 100$

$(\forall n \geq 2)[a_n = a_{n-1} + 3a_{n-2}]$

Find INTEGERS A, B such that

$(\forall n \geq 0)[a_n \leq AB^n]$.

Try to make B as small as possible .

4. (25 points) (This is the Honors Problem from HW04 BUT you will now do it rigorously with induction!)

By the *Four Color Theorem* every planar graph is 4 colorable. You may use this.

A graph has *crossing number c* if there is a way to draw it so that if you REMOVE c edges, the graph is planar.

Prove the following by induction. So give a Base Case, an Induction Hypothesis, and an Induction step.

Every graph of crossing number c is $c + 4$-colorable.

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5. (25 points) In this problem we guide you through a proof that number of the form $4^n(8k + 7)$ cannot be written as the sum of 3 squares.

(a) (0 points but you will need this for later) Find the following set

$$X = \{x^2 \pmod 8 \colon x \in \{0, 1, 2, 3, 4, 5, 6, 7\}\}.$$

(b) (0 points, this is more of a review of what you've already seen) Show that if $x \equiv 7 \pmod 8$ then x is NOT the sum of three squares. Also, we need this later for the base case.

(c) (10 points) Prove that if $x^2 + y^2 + z^2 \equiv 0 \pmod 4$ then x, y, z are all even.

    Hint: Prove the contrapositive.

(d) (15 points) Prove the following by induction on n.

    Theorem Let $n \geq 0$. Let $k \in \mathbb{N}$. Show that $4^n(8k + 7)$ cannot be written as the sum of 3 squares.

    (Hint: Use Part 3.)

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6. (0 points Extra Credit)

   (a) Give a complete prove of the following theorem:

   $\{n : n$ *is prime and* $n \equiv 1 \pmod 4\}$ *is infinite.*

   The proof must be understandable to the students taking 250H. See next item.

   Permission I doubt you can do this without looking it up, so feel free to look it up. The important this is that you *understand* the proof and can explain it.

   (b) Make up slides in LaTeX to prove the theorem above. I may ask you to present it to the class. (If you don't want to, then I might.)