

Questions from Past 250 MIDTERMS

1. In this problem (1) all symbols have their usual meaning, and (2) a domain is a subset of \mathbb{R} .

I) Consider the sentence

$$(A) \quad (\forall x)(\exists y)[x + y = 0].$$

- a) Give an infinite domain where A is TRUE OR prove there is no infinite domain where A is TRUE.
- b) Give an infinite domain where A is FALSE OR prove there is no infinite domain where A is FALSE.
- c) Give a finite domain with **at least three elements** where A is TRUE OR prove there is no finite domain with at least three elements where A is TRUE.
- d) Give a finite domain with **at least three elements** where A is FALSE OR prove there is no finite domain with at least three elements where A is FALSE.

II) Consider the sentence

$$(B) \quad (\forall x)(\exists y)[xy = 1].$$

a) Give an infinite domain where B is TRUE OR prove there is no infinite domain where B is TRUE.

b) Give an infinite domain where B is FALSE OR prove there is no infinite domain where B is FALSE.

c) Give a finite domain with **at least three elements** where B is TRUE OR prove there is no finite domain with at least three elements where B is TRUE.

d) Give a finite domain with **at least three elements** where B is FALSE OR prove there is no finite domain with at least three elements where B is FALSE.

2. Consider the following boolean function:

$$f(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = \begin{cases} T & \text{if exactly ONE of the inputs is T} \\ F & \text{otherwise} \end{cases} \quad (1)$$

- (a) How many rows are in the truth table for f ?
- (b) How many rows in the truth table for f evaluate to TRUE?
- (c) Write down all of the rows that make f evaluate to TRUE.
- (d) Write a formula for f . (Do not write a truth table to do this.)
- (e) Let

$$f_n(x_1, \dots, x_n) = \begin{cases} T & \text{if exactly ONE of the inputs is T} \\ F & \text{otherwise} \end{cases} \quad (2)$$

Name a function g such that the following is true:

The technique used in problem d can be extended to show that there is a formula for f_n of length $O(g(n))$.

3. In this problem the domain is the natural numbers and the language has the usual logical symbols and arithmetic operations.
- (a) A number is *cool* if it can be written as the sum of ≤ 3 cubes. Let $COOL(x)$ mean that x is cool. Write a formula for $COOL(x)$.
 - (b) (You may use $COOL(x)$ in this problem.) Write a sentence to express the following: *There exists an infinite number of numbers that are NOT cool*
 - (c) Give 2 examples of cool numbers. Prove that they are cool.
 - (d) Give 2 examples of numbers that are not cool. Prove that they are not cool.

4. (20 points)

- (a) Compute the following mod 16: $0^4, 1^4, 2^4, \dots, 15^4$.
(You may use the following shortcut: $(16 - a)^4 \equiv a^4 \pmod{16}$.)
- (b) Use the results of part a to find a number N and an infinite set X such that the following is true

If $x \in X$ then x cannot be written as the sum of N fourth powers.

Make X as large as possible.

5. Let $T(n)$ be defined by $T(1) = 0$

$$(\forall n \geq 1) \left[T(n) = T\left(\left\lfloor \frac{n}{11} \right\rfloor\right) + T\left(\left\lfloor \frac{2n}{11} \right\rfloor\right) + T\left(\left\lfloor \frac{3n}{11} \right\rfloor\right) + 2n \right]$$

Use constructive induction to find a constant $A \in \mathbf{N}$ such that

$$(\forall n \geq 0) [T(n) \leq An].$$

6. (a) (0 points but it will be helpful) Compute all of the following mod 7:
 $2^0, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6$
- (b) State a true conjecture about the value of $2^n \pmod{7}$. It should be of the form (and this is NOT true)

$$2^n \pmod{7} = \begin{cases} 1 & \text{if } n \equiv 0, 1 \pmod{5} \\ 4 & \text{if } n \equiv 2, 3 \pmod{5} \\ 6 & \text{if } n \equiv 4 \pmod{5} \end{cases} \quad (3)$$

- (c) Prove your conjecture by induction.