Homework 06, MORALLY Due March 24

1. (0 points) What is your name.

GO TO THE NEXT PAGE

2. (25 points) In this problem we will guide you through the proof that the following set is infinite

$$X = \{p \colon p \equiv 3 \pmod 4 \text{ and } p \text{ is prime } \}.$$

We begin the proof for you:

Assume X is finite. So let $X = \{p_1, \ldots, p_k\}$.

Let

$N = 4p_1p_2 \cdots p_k + 3$

If N is prime then you have found a prime NOT on the list. You are done!

If N is not prime then let $N = q_1^{a_1} \cdots q_m^{a_m}$. where $q_1, \ldots, q_m$ are primes.

YOU NEED TO SHOW THAT at least one of the q_i is $\equiv 3 \pmod 4$.

GO TO NEXT PAGE

3. (25 points) Let a_n be defined as follows

$a_0 = 1$

$a_1 = 100$

$(\forall n \ge 2)[a_n = a_{n-1} + 3a_{n-2}]$

Find INTEGERS A, B such that

$(\forall n \ge 0)[a_n \le AB^n]$.

Try to make B as small as possible .

SOLUTION

We use constructive induction.

Base Cases

$a_0 = 100$. Hence we need $1 \le AB^0 = A$. So $A \ge 1$.

$a_1 = 5$. Hence we need $100 \le AB$. So $AB \ge 100$.

IH For all $n' < n$, $a_{n'} \le AB^{n'}$.

IS

$a_n = a_{n-1} + 3a_{n-2} \le AB^{n-1} + 3AB^{n-2}$.

Hence we want

$AB^{n-1} + 4AB^{n-2} \le AB^n$.

Divide by AB^{n-2} to get

$B + 4 \le B^2$.

Trial and error shows that $B = 3$ works.

So we take $B = 3$.

The constraints on A are $A \ge 1$ and $AB \ge 100$.

The last constraint is now $3A \ge 10$.

$A = 4$ satisfies both constraints.

Final answer: $A = 4$, $B = 3$.

END OF SOLUTION

GO TO NEXT PAGE

4. (25 points) (This is the Honors Problem from HW04 BUT you will now do it rigorously with induction!)

   By the *Four Color Theorem* every planar graph is 4 colorable. You may use this.

   A graph has *crossing number* c if there is a way to draw it so that if you REMOVE c edges, the graph is planar.

   Prove the following by induction (so give a Base Case, an Induction Hypothesis, and an Induction

   Every graph of crossing number c *is* $c+4$*-colorable.*

GO TO NEXT PAGE

5. (25 points) In this problem we guide you through a proof that number of the form $4^n(8k+7)$ cannot be written as the sum of 3 squares.

   (a) (0 points but you will need this for later) Find the following set

   $$X = \{x^2 \pmod 8 \colon x \in \{0,1,2,3,4,5,6,7\}\}.$$

   (b) (0 points, this is more of a review of what you've already seen) Show that if $x \equiv 7 \pmod 8$ then x is NOT the sum of three squares. Also, we need this later for the base case.

   (c) (10 points) Prove that if $x^2 + y^2 + z^2 \equiv 0 \pmod 4$ then x, y, z are all even.

   Hint: Prove the contrapositive.

   SOLUTION

   Part 1:

   The contrapositive is

   if at least one of x, y, z is odd then $x^2 + y^2 + z^2 \not\equiv 0 \pmod 4$.

   We will use:

   if $x \equiv 0 \pmod 2$ then $x^2 \equiv 0 \pmod 4$.

   if $x \equiv 1 \pmod 2$ then $x^2 \equiv 1 \pmod 4$.

   Case 1: Exactly 1 of $\{x, y, z\}$ is odd. We can assume

   $x \equiv 1 \pmod 2$, $y \equiv 0 \pmod 2$, $z \equiv 0 \pmod 2$.

   $x^2 + y^2 + z^2 \equiv 1 + 0 + 0 \equiv 1 \pmod 4$.

   Case 2: Exactly 2 of $\{x, y, z\}$ is odd. We can assume

   $x \equiv 1 \pmod 2$, $y \equiv 1 \pmod 2$, $z \equiv 0 \pmod 2$.

   $x^2 + y^2 + z^2 \equiv 1 + 1 + 0 \equiv 2 \pmod 4$.

   Case 3: Exactly 3 of $\{x, y, z\}$ is odd. We can assume

   $x \equiv 1 \pmod 2$, $y \equiv 1 \pmod 2$, $z \equiv 1 \pmod 2$.

   $x^2 + y^2 + z^2 \equiv 1 + 1 + 1 \equiv 3 \pmod 4$.

   END OF SOLUTION

   (d) (15 points) Prove the following by induction on n.

   Theorem Let $n \geq 0$. Let $k \in \mathbb{N}$. Show that $4^n(8k+7)$ cannot be written as the sum of 3 squares.

   (Hint: Use Part 3.)

SOLUTION

BC $n = 0$. We need that $8k + 7$ is never the sum of 3 squares. This was part 2.

IH $4^n(8k + 7)$ cannot be written as the sum of 3 squares.

IS We can assume $n \geq 1$.

Assume, BWOC, that $4^{n+1}(8k + 7) = x^2 + y^2 + z^2$.

By Part 3 there exists a, b, c such that $x = 2a$, $y = 2b$, $z = 2c$.

Hence $4^{n+1}(8k + 7) = (2a)^2 + (2b)^2 + (2c)^2 = 4a^2 + 4b^2 + 4c^2$.

Divide by 4 to get

$4^n(8k + 7) = a^2 + b^2 + c^2$.

This contradicts the IH. Hence $4^{n+1}(8k + 7)$ cannot be written as the sum of 3 squares.

END OF SOLUTION

GO TO NEXT PAGE

6. (0 points Extra Credit)

   (a) Give a complete prove of the following theorem:

   $\{n\colon n$ *is prime and* $n \equiv 1 \pmod 4\}$ *is infinite.*

   The proof must be understandable to the students taking 250H. See next item.

   Permission I doubt you can do this without looking it up, so feel free to look it up. The important this is that you *understand* the proof and can explain it.

   (b) Make up slides in LaTeX to prove the theorem above. I may ask you to present it to the class. (If you don't want to, then I might.)