Questions from Past 250 MIDTERMS

1. In this problem (1) all symbols have their usual meaning, and (2) a domain is a subset of R.

   I) Consider the sentence

   (A) $$(\forall x)(\exists y)[x + y = 0].$$

   a) Give an infinite domain where A is TRUE OR prove there is no infinite domain where A is TRUE.

   b) Give an infinite domain where A is FALSE OR prove there is no infinite domain where A is FALSE.

   c) Give a finite domain with **at least three elements** where A is TRUE OR prove there is no finite domain with at least three elements where A is TRUE.

   d) Give a finite domain with **at least three elements** where A is FALSE OR prove there is no finite domain with at least three elements where A is FALSE.

II) Consider the sentence

(B) $(\forall x)(\exists y)[xy = 1]$.

a) Give an infinite domain where B is TRUE OR prove there is no infinite domain where B is TRUE.

b) Give an infinite domain where B is FALSE OR prove there is no infinite domain where B is FALSE.

c) Give a finite domain with **at least three elements** where B is TRUE OR prove there is no finite domain with at least three elements where B is TRUE.

d) Give a finite domain with **at least three elements** where B is FALSE OR prove there is no finite domain with at least three elements where B is FALSE.

2. Consider the following boolean function:

$$f(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = \begin{cases} T & \text{if exactly ONE of the inputs is T} \\ F & \text{otherwise} \end{cases} \tag{1}$$

(a) How many rows are in the truthtable for f?

(b) How many rows in the truthtable for f evalute to TRUE?

(c) Write down all of the rows that make f evaluate to TRUE.

(d) Write a formula for f. (Do not write a truth table to do this.)

(e) Let

$$f_n(x_1, \ldots, x_n) = \begin{cases} T & \text{if exactly ONE of the inputs is T} \\ F & \text{otherwise} \end{cases} \tag{2}$$

Name a function g such that the following is true:
The technique used in problem d can be extended to show that there is a formula for f_n of length $O(g(n))$.

3. In this problem the domain is the natural numbers and the language has the usual logical symbols and arithmetic operations.

   (a) A number is *cool* if it can be written as the sum of ≤ 3 cubes. Let $COOL(x)$ mean that x is cool. Write a formula for $COOL(x)$.

   (b) (You may use $COOL(x)$ in this problem.) Write a sentence to express the following: *There exists an infinite number of numbers that are NOT cool*

   (c) Give 2 examples of cool numbers. Prove that they are cool.

   (d) Give 2 examples of numbers that are not cool. Prove that they are not cool.

4. (20 points)

   (a) Compute the following mod 16: $0^4, 1^4, 2^4, \ldots, 15^4$.
   (You may use the following shortcut: $(16-a)^4 \equiv a^6 \pmod{16}$.)

   (b) Use the results of part a to find a number N and an infinite set X such that the following is true

   If $x \in X$ then x cannot be written as the sum of N fourth powers.

   Make X as large as possible.

5. Let $T(n)$ be defined by $T(1) = 0$

$$(\forall n \geq 1)\left[T(n) = T\left(\left\lfloor \frac{n}{11} \right\rfloor\right) + T\left(\left\lfloor \frac{2n}{11} \right\rfloor\right) + T\left(\left\lfloor \frac{3n}{11} \right\rfloor\right) + 2n\right]$$

Use constructive induction to find a constant $A \in \mathsf{N}$ such that

$$(\forall n \geq 0)[T(n) \leq An].$$

6. (a) (0 points but it will be helpfu) Compute all of the following mod 7:
2^0, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6

(b) State a true conjecture about the value of $2^n \pmod 7$. It should be of the form (and this is NOT true)

$$2^n \bmod 7 = \begin{cases} 1 & \text{if } n \equiv 0, 1 \pmod 5 \\ 4 & \text{if } n \equiv 2, 3 \pmod 5 \\ 6 & \text{if } n \equiv 4 \pmod 5 \end{cases} \tag{3}$$

(c) Prove your conjecture by induction.