Discussion Slides: HW 09 Solutions and Catalan Numbers

Horse Numbers, Counting, Stars and Bars, and Full Binary Trees

CMSC250H Discussion

HW 09 Problem 2: Race outcomes with $x_1 < x_n$

Let $H(n)$ be the number of race outcomes of n horses when ties are allowed. Let $I(n)$ count the outcomes with $x_1 < x_n$.

Key symmetry

For every outcome, exactly one of these happens:

$$x_1 < x_n, \qquad x_1 = x_n, \qquad x_n < x_1.$$

By symmetry, the first and third cases have the same count: $I(n)$ each.

If $x_1 = x_n$, merge horses x_1 and x_n into one tied object, giving $H(n-1)$ outcomes.

$$\boxed{H(n) = 2I(n) + H(n-1)} \qquad \Rightarrow \qquad \boxed{I(n) = \frac{H(n) - H(n-1)}{2}}.$$

HW 09 Problem 2: A recurrence using Horse Numbers

The Horse Numbers satisfy

$$H(0) = 1, \qquad H(n) = \sum_{j=1}^{n} \binom{n}{j} H(n-j),$$

by choosing the first-place tie class of size j.
Therefore, for $n \geq 2$,

$$\boxed{I(n) = \frac{1}{2}\left(\sum_{j=1}^{n} \binom{n}{j} H(n-j) - H(n-1)\right)}.$$

Small values

$$H(1) = 1, \quad H(2) = 3, \quad H(3) = 13, \quad H(4) = 75.$$

$$\boxed{I(2) = 1, \qquad I(3) = 5, \qquad I(4) = 31.}$$

HW 09 Problem 3: Inclusion-exclusion

Let G, R, A be the sets of students taking Game Theory, Ramsey Theory, and AI.

$$|G \cup R \cup A| = |G| + |R| + |A| - |G \cap R| - |G \cap A| - |R \cap A| + |G \cap R \cap A|.$$

$$|G \cup R \cup A| = 30 + 30 + 30 - 10 - 10 - 10 + 5 = \boxed{65}.$$

Game Theory and Ramsey Theory, but not AI

$$|G \cap R \cap A^c| = |G \cap R| - |G \cap R \cap A| = 10 - 5 = \boxed{5}.$$

HW 09 Problem 4: Stars and bars

For nonnegative integer solutions to

$$w + x + y + z = N,$$

the number of solutions is

$$\binom{N+4-1}{4-1} = \binom{N+3}{3}.$$

(a) $w, x, y, z \geq 0$:

$$\binom{103}{3} = \boxed{176{,}851}.$$

(b) $w, x, y, z \geq 5$: set $w' = w - 5,\ x' = x - 5,\ y' = y - 5,\ z' = z - 5$. Then $w' + x' + y' + z' = 80$.

$$\binom{83}{3} = \boxed{91{,}881}.$$

HW 09 Problem 4: Part (c)

For part (c), we need

$$w + x + y + z = 100, \qquad w \geq 1, \qquad x \geq 10.$$

Set

$$w' = w - 1, \qquad x' = x - 10.$$

Then

$$w' + x' + y + z = 100 - 1 - 10 = 89.$$

So the number of solutions is

$$\binom{89 + 4 - 1}{4 - 1} = \binom{92}{3} = \boxed{125{,}580}.$$

Catalan numbers: definition and formula

A Dyck path in an $n \times n$ grid is a lattice path from $(0,0)$ to (n,n) that never goes above $y = x$.

Catalan number

C_n is the number of Dyck paths in the $n \times n$ grid.

Using the reflection argument for bad paths:

$$\boxed{C_n = \binom{2n}{n} - \binom{2n}{n-1}}$$

Equivalently,

$$\boxed{C_n = \frac{1}{n+1}\binom{2n}{n}}.$$

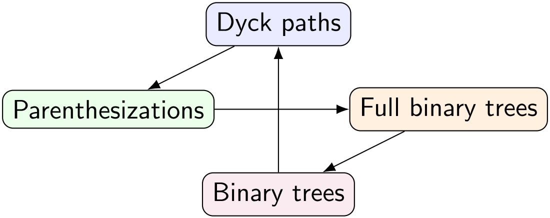
First values

$$C_0 = 1, \quad C_1 = 1, \quad C_2 = 2, \quad C_3 = 5, \quad C_4 = 14, \quad C_5 = 42.$$

Catalan objects that appeared in the reference

Catalan numbers count many different objects. The reference connects C_n to:

- Dyck paths in an $n \times n$ grid
- Ballot sequences of length $2n$
- Parenthesizations of $n+1$ letters
- Noncrossing arcs on $2n$ points
- Nonintersecting chords on $2n$ points
- Binary trees on n vertices
- Full binary trees on $2n+1$ vertices
- Triangulations of polygons

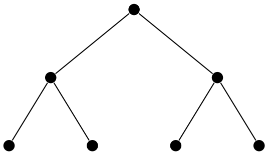


Full binary trees

A **binary tree** is recursively built from a root, a left subtree, and a right subtree.
A **full binary tree** is a binary tree where every vertex has either:

0 children or 2 children.

- Internal vertices represent binary choices or products.
- Leaves represent final outputs or letters.
- A full binary tree with n internal vertices has $n+1$ leaves and $2n+1$ total vertices.



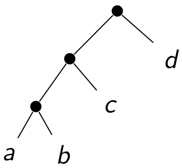
Bijection: parenthesizations $\leftrightarrow$ full binary trees

A parenthesization of $n+1$ letters tells us the order of multiplying adjacent terms.

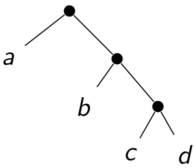
Map to a full binary tree

- Each multiplication becomes an internal vertex.
- The two factors being multiplied become the left and right subtrees.
- Each letter becomes a leaf.

$((ab)c)d$



$a(b(cd))$



Bijection: Parenthesizations $\leftrightarrow$ Dyck Paths

Claim: The number of parenthesizations of $n+1$ symbols is C_n.

Proof.

Suppose we have a parenthization of $n+1$ letters. This corresponds to an outer operation, and $n-1$ inner operations $\implies$ $2n-2$ parentheses.

So our string has $n+1+2n-2 = 3n-1$ total symbols

Remove each of the right parentheses and the last letter to get a new string with $2n-1$ symbols. Notice that we can just reconstruct the original string by looking at the left parentheses relationships to the letters. Now, add one (at the start. Replace each (with a U and each letter with an R. This gives us a valid walk, since we always start with U and never have more $R's$ than $U's$ by the definition of a valid parenthesizations. $\square$

Why the count is C_n

Since parenthesizations of $n+1$ letters are in bijection with full binary trees having:

$$n \text{ internal vertices}, \qquad n+1 \text{ leaves}, \qquad 2n+1 \text{ total vertices},$$

we get:

$$\boxed{\#\{\text{full binary trees on } 2n+1 \text{ vertices}\} = C_n.}$$

For $n = 3$

There are

$$C_3 = 5$$

full binary trees on $2(3)+1 = 7$ vertices.

Bijection: full binary trees $\leftrightarrow$ binary trees

There is also a direct Catalan bijection:

$$\boxed{\text{full binary trees on } 2n+1 \text{ vertices}} \longleftrightarrow \boxed{\text{binary trees on } n \text{ vertices}}.$$

One clean construction

Given a binary tree with n vertices:

- Keep each original vertex as an internal vertex.
- If a left child is missing, attach a new leaf in that left position.
- If a right child is missing, attach a new leaf in that right position.

The result is a full binary tree with n internal vertices and $n+1$ leaves.

Reverse the map by deleting all leaves from the full binary tree.