

Horse Numbers

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$$H(3) = 13$$

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$$H(n) = \sum_{i=1}^n \binom{n}{n-i} H(n-i) = \sum_{i=0}^{n-1} \binom{n}{i} H(i).$$