Homework 6 Solutions

CMSC250H

Problem 2

Question

Give a complete proof that the following set is infinite:

$$\{n\colon\ n \text{ is prime and } n \equiv 3 \pmod 4\}$$

Problem 2

Proof.

Suppose for contradiction that there are a finite number of primes congruent to $3 \pmod 4$. Let

$$\{p_1, p_2, \ldots, p_k\}$$

denote the finite set of primes congruent to $3 \pmod 4$. Consider the number

$$N = 4(p_1 p_2 p_3 \ldots p_k) - 1$$

☐

Problem 2

Proof.

Clearly, N is congruent to

$$3 \mod 4$$

(since $-1 \equiv 3 \mod 4$)

If it is prime, then we have a contradiction, since this is a new prime not in our previous set due to the fact no p_i is a factor.

Now, suppose N is composite, and let

$$N = q_1^{a_1} \dots q_m^{a_m}$$

where all q_i are prime. ☐

Problem 2

Proof.

If all q_i are $1 \pmod 4$, N itself must be $1 \pmod 4$.

If any were $2 \mod 4$, then N would be even.

Thus, N has a prime factor q_i such that $q_i \equiv 3 \pmod 4$. Since no q_i divides N, no p_i divides q_i.

Hence, q_i is a prime $\equiv 3 \pmod 4$ that is not in our set, a contradiction. ☐

Problem 3

Question

Let a_n be defined as follows

$a_0 = 1$

$a_1 = 100$

$(\forall n \ge 2)[a_n = a_{n-1} + 3a_{n-2}]$

Find INTEGERS A, B such that $(\forall n \ge 0)[a_n \le AB^n]$
Try to make B as small as possible .

Problem3

Proof.

We use constructive induction:

Base Cases:
$a_0 = 1$. Hence we need $1 \leq AB^0 = A$. So $A \geq 1$.

$a_1 = 100$. Hence we need $100 \leq AB$. So $AB \geq 100$.

IH: For all $n' < n$, $a_{n'} \leq AB^{n'}$.

☐

Frame Title

Proof.

IS: $a_n = a_{n-1} + 3a_{n-2} \leq AB^{n-1} + 3AB^{n-2}$.

Hence we want

$$AB^{n-1} + 3AB^{n-2} \leq AB^n$$

Divide by AB^{n-2} to get

$$B + 3 \leq B^2$$

Trial and error shows that $B = 3$ works.
So we take $B = 3$.
The constraints on A are $A \geq 1$ and $AB \geq 100$.
The last constraint is now $3A \geq 100$.
$A = 34$ satisfies both constraints.
Final answer:

$$A = 34, B = 3$$

Problem 4

Question

Every graph of crossing number c is $c + 4$-colorable.

Problem 4

Proof.

We induct on the crossing number c.

Base Case ($c = 0$) : For a graph G with $c = 0$, it is planar and therefore 4-colorable, so the statement holds.

Induction Hypothesis: Assume it holds true for graphs of crossing number $c - 1$. ☐

Problem 4

Proof.

IS: Consider a graph with crossing number c.

Remove an edge that crosses over any other edge.

By the IH, we know have a graph with crossing number $c - 1$, so this graph is now 4+$c - 1$ colorable.

Add the edge back to obtain the original graph.

The only case to consider is if both end points have the same color. If both endpoints of the re-added edge are the same color, we can now simply change one of their colors to make it a proper coloring. Hence, the graph is $4 + c$ colorable. ☐

Problem 5c

Question

Prove that if $x^2 + y^2 + z^2 \equiv 0 \pmod{4}$ then x, y, z are all even.

Problem 5c

Proof.

The contrapositive is
if at least one of x, y, z is odd then

$$x^2 + y^2 + z^2 \not\equiv 0 \pmod{4}$$

We will use:

$$\text{if } x \equiv 0 \pmod{2} \text{ then } x^2 \equiv 0 \pmod{4}$$

$$\text{if } x \equiv 1 \pmod{2} \text{ then } x^2 \equiv 1 \pmod{4}$$

☐

Problem 5c

Proof.

Case 1: Exactly 1 of $\{x, y, z\}$ is odd. We can assume

$$x \equiv 1 \pmod{2} \qquad y \equiv 0 \pmod{2} \qquad z \equiv 0 \pmod{2}$$

$$x^2 + y^2 + z^2 \equiv 1 + 0 + 0 \equiv 1 \pmod{4}$$

Case 2: Exactly 2 of $\{x, y, z\}$ is odd. We can assume

$$x \equiv 1 \pmod{2} \qquad y \equiv 1 \pmod{2} \qquad z \equiv 0 \pmod{2}$$

.

$$x^2 + y^2 + z^2 \equiv 1 + 1 + 0 \equiv 2 \pmod{4}$$

□

Problem 5c

Proof.

Case 3: Exactly 3 of $\{x, y, z\}$ is odd. We can assume

$$x \equiv 1 \pmod{2} \qquad y \equiv 1 \pmod{2} \qquad z \equiv 1 \pmod{2}$$

$$x^2 + y^2 + z^2 \equiv 1 + 1 + 1 \equiv 3 \pmod{4}$$

☐

Problem 5d

Question

Let $n \geq 0$. Let $k \in$. Show that

$$4^n(8k+7)$$

cannot be written as the sum of 3 squares.

Problem 5d

Proof.

BC: $n = 0$. We need that $8k + 7$ is never the sum of 3 squares. This was part b).

IH: $4^n(8k + 7)$ cannot be written as the sum of 3 squares.

IS: We can assume $n \geq 1$.
Assume, BWOC, that

$$4^{n+1}(8k + 7) = x^2 + y^2 + z^2$$

By Part 3 there exists a, b, c such that $x = 2a$, $y = 2b$, $z = 2c$. Hence

$$4^{n+1}(8k + 7) = (2a)^2 + (2b)^2 + (2c)^2 = 4a^2 + 4b^2 + 4c^2$$

. □

Problem 5d

Proof.

Divide by 4 to get

$$4^n(8k+7) = a^2 + b^2 + c^2$$

This contradicts the IH. Hence

$$4^{n+1}(8k+7)$$

cannot be written as the sum of 3 squares. □