Strong Induction and Inequalities

Nice Recurrences

In the Strong Induction Slide Packet we studied the recurrence

$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ 8 & \text{if } n = 1 \\ a_{n-1} + 2a_{n-2} & \text{if } n \geq 2 \end{cases} \tag{1}$$

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Does This Recurrence Have a Nice Closed Form?

$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ 3 & \text{if } n = 2 \\ a_{n-1} + 11a_{n-2} + 13a_{n-3} & \text{if } n \geq 2 \end{cases} \qquad (3)$$

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There is a closed form on the next slide.

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$g(n) =$

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Finish on next slide.

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$$93 \le 125 \text{ TRUE!}$$

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(5) This is called **Constructive Induction**. It's the topic of the next slide packet.