

CMSC 250H - More Induction/Contradiction Proofs

March 4, 2026

What is e ?

The number e is defined by the infinite series:

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \dots$$

Numerically

$$e \approx 2.71828182845\dots$$

Goal

Prove that e is **irrational**, i.e. $e \neq \frac{p}{q}$ for any integers p, q .

This proof is due to **Joseph Fourier** (c. 1815) and is one of the most elegant irrationality arguments in mathematics.

Proof by Contradiction

Assume for the sake of contradiction that e is rational:

$$e = \frac{p}{q}, \quad p, q \in \mathbb{Z}, \quad q > 0.$$

The key trick: multiply both sides by $q!$ and examine what remains after subtracting the first $q + 1$ terms of the series.

We will construct an integer N satisfying $0 < N < 1$ — a contradiction.

Step 1: Define the Remainder

Write e as its series and split at the q -th term:

$$e = \underbrace{\sum_{k=0}^q \frac{1}{k!}}_{\text{partial sum}} + \underbrace{\sum_{k=q+1}^{\infty} \frac{1}{k!}}_{\text{tail } R_q}$$

Define N

$$N = q! \left(e - \sum_{k=0}^q \frac{1}{k!} \right) = q! \cdot R_q$$

We will show:

- 1 N is a **positive integer** (if $e = p/q$), and
- 2 $N < 1$.

These two facts together give a contradiction.

Step 2: N is an Integer (assuming $e = p/q$)

Expanding N :

$$N = q! \cdot e - q! \sum_{k=0}^q \frac{1}{k!}$$

First term

If $e = p/q$, then $q! \cdot e = q! \cdot \frac{p}{q} = (q-1)!p \in \mathbb{Z}$.

Second term

$$q! \sum_{k=0}^q \frac{1}{k!} = \sum_{k=0}^q \frac{q!}{k!}$$

For each $0 \leq k \leq q$, the term $\frac{q!}{k!} = q(q-1)\cdots(k+1) \in \mathbb{Z}$.

Therefore N is the difference of two integers: $N \in \mathbb{Z}$.

Step 3: $N > 0$

Recall:

$$N = q! \cdot R_q = q! \sum_{k=q+1}^{\infty} \frac{1}{k!}$$

Since every term of the series is **positive**:

$$R_q = \sum_{k=q+1}^{\infty} \frac{1}{k!} > 0$$

Therefore:

$$\boxed{N > 0}$$

Since N is an integer and $N > 0$, we have $N \geq 1$.

Now we just need to show $N < 1$ to get our contradiction.

Step 4: $N < 1$

We bound R_q from above. For $k \geq q + 1$:

$$k! = (q + 1)(q + 2) \cdots k \cdot q! \geq (q + 1)^{k-q} \cdot q!$$

Therefore each term satisfies:

$$\frac{1}{k!} \leq \frac{1}{q! \cdot (q + 1)^{k-q}}$$

Summing:

$$R_q \leq \frac{1}{q!} \sum_{j=1}^{\infty} \frac{1}{(q + 1)^j} = \frac{1}{q!} \cdot \frac{1/(q + 1)}{1 - 1/(q + 1)} = \frac{1}{q!} \cdot \frac{1}{q}$$

So:

$$N = q! \cdot R_q \leq q! \cdot \frac{1}{q \cdot q!} = \frac{1}{q} \leq 1$$

Step 4 (continued): Strict Inequality

We showed:

$$N \leq \frac{1}{q}$$

For $q \geq 2$ (the only interesting case, since $e \approx 2.718$ is not an integer, ruling out $q = 1$):

$$N \leq \frac{1}{q} \leq \frac{1}{2} < 1$$

Summary of the bound

$$0 < N < 1$$

But we established that N is an integer. There is **no integer** strictly between 0 and 1.

The Contradiction

Let us collect what we have shown, *assuming* $e = p/q$:

- 1 $N = q! \left(e - \sum_{k=0}^q \frac{1}{k!} \right)$ is an integer.
- 2 $N > 0$ (tail of a positive series).
- 3 $N < 1$ (geometric series bound gives $N \leq 1/q \leq 1/2$).

Contradiction

There is no integer N with $0 < N < 1$. Our assumption that $e = p/q$ must be **false**.

Conclusion

Theorem (Fourier, c. 1815)

e is irrational.

Key ideas in this proof:

- Express e as a convergent series $\sum_{k=0}^{\infty} \frac{1}{k!}$.
- Multiply by $q!$ to force integrality of the truncated part.
- Bound the tail using a geometric series comparison.
- Reach a contradiction: an integer strictly between 0 and 1.

A stronger result

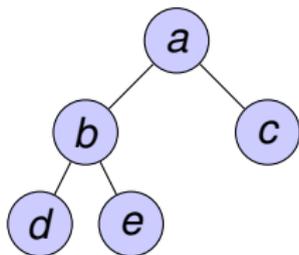
Hermite (1873) proved the harder fact that e is **transcendental** (not a root of any polynomial with rational coefficients).

What is a Tree?

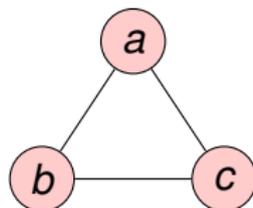
Definition: Tree

A **tree** is a connected, acyclic (undirected) graph.

Examples of trees:

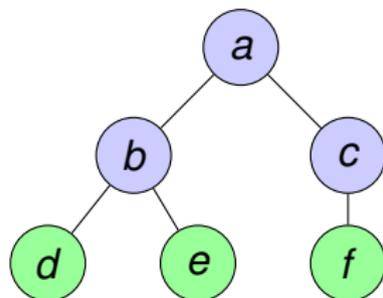


NOT a tree (has a cycle):



Definition: Leaf

A **leaf** is a vertex of degree 1 (exactly one edge incident to it).



Leaves (degree 1, shaded green): d, e, f .

Internal nodes (degree ≥ 2 , shaded blue): a, b, c .

The Theorem

Theorem

Let G be a tree with $n \geq 1$ vertices. Then G has exactly $n - 1$ edges.

Proof strategy

By **strong induction** on n , using the following key lemma:

Key Lemma (Two Leaves)

Every tree with $n \geq 2$ vertices has **at least two leaves**.

We will:

- 1 Prove the Key Lemma.
- 2 Use it to do the inductive step: remove a leaf, apply the IH, reattach.

Proving the Key Lemma

Claim

Every tree G with $n \geq 2$ vertices has at least two leaves.

Proof.

Consider a **longest path** in G :

$$v_1 - v_2 - \cdots - v_k$$

Since $n \geq 2$ and G is connected, such a path has $k \geq 2$.

Claim: v_1 and v_k are both leaves.

Suppose v_1 has a neighbor $u \neq v_2$.

- If $u \notin \{v_1, \dots, v_k\}$: the path $u - v_1 - \cdots - v_k$ is longer. Contradiction.
- If $u = v_i$ for some $i \geq 3$: the walk $v_1 - v_2 - \cdots - v_i - v_1$ forms a cycle. Contradicts G being a tree.

Main Proof: Base Case

Theorem

A tree on n vertices has $n - 1$ edges.

Proof.

By strong induction on n .

Base Case ($n = 1$):

A single vertex with no edges.

$$\text{edges} = 0 = 1 - 1. \checkmark$$



Main Proof: Inductive Hypothesis

Proof (continued).

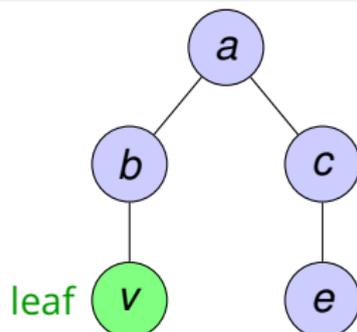
Inductive Hypothesis (IH):

For all $1 \leq m < n$, every tree on m vertices has $m - 1$ edges.

Inductive Step:

Let G be a tree on $n \geq 2$ vertices. We want to show G has $n - 1$ edges.

By the **Key Lemma**, G has at least two leaves. Pick any one leaf v . □



Main Proof: The Removal Step

Proof (continued).

Let $G' = G \setminus \{v\}$ (remove leaf v and its single edge).

G' is a tree:

- *Acyclic*: removing a vertex cannot create a cycle. ✓
- *Connected*: v was a leaf, so removing it cannot disconnect the rest. ✓

G' has $n - 1$ vertices, so by the **IH**:

$$|E(G')| = (n - 1) - 1 = n - 2.$$

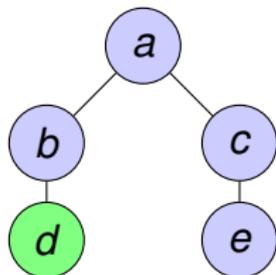
Reattaching v adds exactly **one edge**, so:

$$|E(G)| = |E(G')| + 1 = (n - 2) + 1 = n - 1. \quad \blacksquare$$



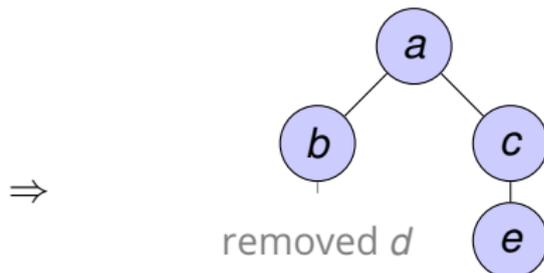
Visualizing the Inductive Step

Tree G ($n = 5$ vertices)



4 edges = $n - 1$

Tree G' ($n - 1 = 4$ vertices)



3 edges = $(n - 1) - 1$

Remove leaf d \rightarrow smaller tree \rightarrow apply IH \rightarrow add 1 back.

Why Do We Need Two Leaves?

You might wonder: why prove there are *two* leaves if we only remove *one*?

Answer

The Key Lemma guarantees **at least one leaf exists** so the inductive step is always possible. Having two gives a stronger result, but one suffices for the proof.

Bonus: the second leaf matters for other results

- Characterizing trees by diameter endpoints.
- Proving trees have no “dead ends” – you can always find two exit points.
- Used in proofs about spanning trees and graph connectivity.