(1) My Curiosity is Bigger than my Ego but...

(2) An Induction Problem where the Base Case is Harder than the Induction Step

A Problem on Reciprocals from a Romanian Math Problem Book

Based on the Blog

A Nice Problem from a Romanian Math Problem Book, Jan 26, 2015

1 Point

Here is the problem we will be discussing:

Prove that for all $n \geq 6$ *there exists* $x_1, \ldots, x_n \in \mathsf{N}$ *such that* $\sum_{i=1}^{n} \frac{1}{x_i^2} = 1$.

This was given to my by by Teaching Assistant for discrete math Ioana Bercea who is a Romanian. She got it out of Romanian math problem book. When given a problem that you have a hard time solving what is bigger your curiosity (so you want to be told the solution) or our ego (so you want to find the solution yourself!). For me it is always my curiosity.

I was initially unable to solve the problem so I looked at the book. The answer was in Romanian! And Ioana was out of town! Now what? I was forced to solve it myself if I wanted to know the answer. So I did!

When I solved it I noticed that the induction step was easy but the base case was hard. Thats unusual, though I have seen it before.

2 Restate the Problem

Def 2.1 Let $Q(n)$ be the statement $(\exists x_1, \ldots, x_n \in \mathsf{N})[\sum_{i=1}^{n} \frac{1}{x_i^2} = 1]$.

Problem: Prove that for all $n \geq 6$ $Q(n)$ is true.

Plan: We will prove (not in this order) (I) $Q(6), Q(7), Q(8)$ and (II) $(\forall n)[Q(n) \implies Q(n+3)]$.

3 Induction Step

Assume we have $Q(i)$ for all $i \in \{6, 7, 8, \ldots, n-1\}$. We show $Q(n)$.

Note that we have $Q(n-3)$ so

$$(\exists x_1,\ldots,x_{n-3} \in \mathsf{N})[\sum_{i=1}^{n} \frac{1}{x_i^2} = 1].$$

Hence $\sum_{i=1}^{n} \frac{1}{(2x_i)^2} = \frac{1}{4}]$.. Therefore

$$\frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^2} + \sum_{i=1}^{n} \frac{1}{(2x_i^2)} = 2.$$

Hence we have $Q(n)$.

4 Base Case

We could write (or get a student to write) a program to prove $Q(6), Q(7), Q(8)$. But what fun would that be? Instead we prove $Q(6), Q(7), Q(8)$ without a program. The proof is somewhat long so its not clear whether a program would have been more fun. As a bonus we get an alternative proof of $Q(n) \implies Q(n+3)$.

Def 4.1 Let $P(n, s, t_e, t_o)$ be YES if there exists a multiset $\{x_1, \ldots, x_n\}$ such that

- $\sum_{i=1}^{n} \frac{1}{x_i^2} = 1$.
- $\{x_1, \ldots, x_n\} = A_1 \cup \cdots \cup A_s \cup L_e \cup L_o$ where all of these multisets are disjoint, each A_i has four of the same even number in them, L_e contains t_e even numbers, L_o contains t_o odd numbers.

Note that we have $P(1, 0, 0, 1)$ via one 1.

Lemma 4.2

1. *If* $t_o \geq 1$ *then* $P(n, s, t_e, t_o) \implies P(n+3, s+1, t_e, t_o - 1)$.
2. *If* $t_e \geq 1$ *then* $P(n, s, t_e, t_o) \implies P(n+3, s+1, t_e, t_o)$.

3. *If* $s \geq 1$ *then* $P(n, s, t_e, t_o) \implies P(n+3, s, t_e+3, t_o)$.

4. $(\forall n \geq 1)[Q(n) \implies Q(n+3)]$ *(This follows from the first three.)*

Proof: 1, (2,3): Replace an $x \in L_o$ ($x \in L_e$, $x \in A_1 \cup \cdots \cup A_s$) with $\{2x, 2x, 2x, 2x\}$. ▮

By Lemma 4.2 and $P(1, 0, 0, 1)$ we get $P(4, 1, 0, 0)$ and then $P(7, 1, 3, 0)$. Hence we get $Q(7)$. We do this explicitly.

$P(1, 0, 0, 1)$ via one 1.

$P(4, 1, 0, 0)$ via four 2's

$P(7, 1, 3, 0)$ via three 2's and four 4's

Lemma 4.3

1. *If* $t_e \geq 1$ *then* $P(n, s, t_e, t_o) \implies P(n+8, s+2, t_e, t_o)$.

2. *If* $t_o \geq 1$ *then* $P(n, s, t_e, t_o) \implies P(n+8, s, t_e, t_o+8)$.

3. *If* $s \geq 1$ *then* $P(n, s, t_e, t_o) \implies P(n+8, s+1, t_e+4, t_o)$.

Proof: 1, (2,3): Replace an $x \in L_e$ ($x \in L_o$, $x \in A_1 \cup \cdots \cup A_s$) with $\{3x, 3x, 3x, 3x, 3x, 3x, 3x, 3x, 3x\}$. ▮

By Lemma 4.3.3 and $P(4, 1, 0, 0)$ we obtain $P(12, 2, 4, 0)$. Then use Lemma 4.3.1 to obtain $P(20, 4, 4, 0)$. We do this explicitly.

$P(4, 1, 0, 0)$ via four 2's.

$P(12, 2, 4, 0)$ via three 2's and nine 6's.

$P(20, 4, 4, 0)$ via three 2's and eight 6's and nine 18's.

Lemma 4.4 *If* $s \geq 1$ *then* $P(n, s, t_e, t_o) \implies P(n-3, s-1, t'_e, t'_o)$ *where exactly one of* t'_e, t'_o *is one more than it was and the other stays the same.*

Proof: Replace $A_1 = \{x, x, x, x\}$ with $\{\frac{x}{2}\}$. (Recall that the A_i's have all even elements.) If $\frac{x}{2}$ is even then t_e increases by one. If $\frac{x}{2}$ is odd then t_o increases by one. ∎

Apply Lemma 4.4 four times to $P(20, 4, 4, 0)$ to obtain $P(8, 0, 4, 0)$, so we have $Q(8)$. We do this explicitly.

$P(20, 4, 4, 0)$ via three 2's and eight 6's and nine 18's.

$P(17, 3, 4, 0)$ via three 2's and eight 6's and one 9 and five 18's

$P(14, 2, 4, 0)$ via three 2's and eight 6's and two 9's and one 18.

$P(11, 1, 4, 0)$ via three 2's and one 3 and five 6's and two 9's and one 18.

$P(8, 0, 4, 0)$ via three 2's and two 3's and one 6 and two 9's and one 18.

Apply Lemma 4.4 twice to $P(12, 2, 4, 0)$ to obtain $P(6, 0, 4, 0)$ so we have $Q(6)$.

$P(12, 2, 4, 0)$ via three 2's and nine 6's.

$P(9, 1, 4, 0)$ via three 2's and one 3 and five 6's.

$P(6, 0, 4, 0)$ via three 2's and two 3 and one 6.

We have $Q(6), Q(7), Q(8)$ and $(\forall n \geq 1)[Q(n) \implies Q(n + 3)]$. Hence we have $(\forall n \geq 6)[Q(n)]$.

Some notes.

1. The solution in the back of the book just gave the numbers to prove $Q(6), Q(7), Q(8)$ and proved $Q(n) \implies Q(n + 3)$. There numbers were

    - $Q(6)$: three 2's two 3's and one 6. Same as mine.
    - $Q(7)$: three 2's and four 4's. Same as mine.
    - $Q(8)$: three 2's, two 3's, one 7, one 14, one 21. Different from mine.

    They do not say how they got it.

2. $Q(5)$ is false by a case by case analysis: You must use AT LEAST three 2's since if you used two 2's and three 3's then you get $2 \times \frac{1}{4} + 3 \times \frac{1}{9} < 1$. Hence we need (a, b) such that

$\frac{1}{a^2} + \frac{1}{b^2} = 1 - \frac{3}{4} = \frac{1}{4}$. We leave it to the reader to show this cannot be done.

3. Note that the theorem with 6 is optimal.

5 A More General Theorem

We can prove a more general theorem but without stating the starting point.

Def 5.1 Let $k \in \mathsf{N}$. Let $Q_k(n)$ be the statement $(\exists x_1, \ldots, x_n \in \mathsf{N})[\sum_{i=1}^{n} \frac{1}{x_i^k} = 1]$.

Theorem 5.2 *For all k there exists n_o such that for all $n \geq n_o$ $Q_k(n)$ is true.*

Proof: Note that $Q_k(1)$ is true as $1 = \frac{1}{1^k}$.

Let $i \in \mathsf{N}$. Clearly $Q_k(n) \implies Q(n + i^k - 1)$: replace $\frac{1}{a_n^k}$ with i^k copies of $\frac{1}{(ia_n)^k}$. Hence for all $x_2, \ldots, x_m$ (any m), if

$$n = 1 + (2^k - 1)x_2 + (3^k - 1)x_3 + \cdots + (m^k - 1)x_k$$

then we have $Q(n)$. It is well known that if $a_1, a_2, \ldots, a_k$ are rel prime then almost all natural numbers can be written as a linear combination of them with positive coefficients. Hence we need to show that some subset of $\{2^k - 1, 3^k - 1, \ldots\}$ is rel prime. Let $d = GCD(2^k - 1, 3^k - 1)$. If $d = 1$ then you are done. If $d \geq 2$ then $GCD(2^k - 1, 3^k - 1, d^k - 1) = 1$ and we are done.

Alternative: $GCD(2^k - 1, 2^{2^k - 1} - 1) = 1$. ∎

6 Open Questions

1. Obtain a less labor-intensive proof of $Q(6), Q(7), Q(8)$ that does not use a computer program.

2. Obtain upper and lower bounds on n_o as a function of k from the Theorem 5.2

3. How hard is the following problem: Given (k, n) determine if 1 can be written a the sum of n inverses-kth-powers. If yes then produce a way to do this. (Greedy does not work— it fails for $k = 2$, $n = 8$.)
4. How hard is the following problem: Given (k, n) determine how many ways 1 can be written as the sum of n inverses-kth-powers.