Reciprocal Theorems

Problem 2 on UMCP HS Math Comp 2010

(a) The equations $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$ and $\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{42}=1$ express 1 as the sum of three (respectively four) distinct positive integers.

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(b) Prove that for any integer $m \geq 3$ there exists m positive integers $d_1<d_2<\cdots<d_m$ such that $\frac{1}{d_1}+\cdots+\frac{1}{d_m}=1$.

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There are 6 ways.

ChatGPT gave me 9 ways but 3 were incorrect.

All ways to Write 1 as Sum of 5 Reciprocals

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How many ways are there to write 1 as the sum of 5 reciprocals?

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Vote

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between 1 and 5 (so less than sums-of-4)

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between 27 and 46

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Answer on the next page

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I chose to believe that.

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160 students got part b correct. There were four different correct solutions. I present those, then some other ones.

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All of the proofs are by induction.

Base Case for All of the Proofs

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Proof One

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Proof Two. Bigger Base Case and $P(n) \rightarrow P(n+2)$

An Induction Scheme

Bill wants to prove $(\forall n \geq 3)[P(n)]$.

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For the case at hand we already did the $n = 3$ and $n = 4$ base case.

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Proof Three. Loading the IH

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Our next proof does this and make some other points of interest.

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Can we use this?

Lets try to use it manually.

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It works so long as the last number is $\equiv 0 \pmod 2$.

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Loading the IH Proving a harder theorem so that the IH is stronger.

IH and IS

IB $d = 3$. $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$, $6 \equiv 0$.

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$$1 = \frac{1}{d_1} + \cdots + \frac{1}{d_n} = \frac{1}{d_1} + \cdots + \frac{1}{d_{n-1}} + \frac{1}{3d_n/2} + \frac{1}{3d_n}.$$

Also NEED that the last number is $\equiv 0$. It is since $3d_n \equiv d_n \equiv 0$.

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Proof Five. Darling's Proof

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Bill: You came up with 2 more proofs of the recip thm:

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Hence $1 = \frac{1}{d_1} + \cdots + \frac{1}{d_{n-1}} + \frac{1}{8d} + \frac{1}{24d}$

Key $24d \equiv 0 \pmod{6}$, so satisfies loaded IH.

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WORK IN GROUPS TO EITHER GET A PROOF USING
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