HW 6 CMSC 452. Morally DUE Mar 14
THIS HOMEWORK IS THREE PAGES
SOLUTIONS

1. (0 points) What is your name? Write it clearly. When is the midterm? Write that clearly too. Staple your HW. WHAT IS THE DAY/TIME OF THE MIDTERM? (HINT: The Midterm is March 30 IN CLASS at 11:00.)

2. (40 points)

   (Presburger Arithmetic convention: a string of 0's and 1's is a number in binary, and we feed it into a DFA with the lower order bits first.)

   (a) (20 points) Write a DFA for

   $$\{x : x \equiv 1 \pmod 3\}$$

   Note that x is in base 2.

   (b) (20 points) Write a DFA for

   $$\{(x, y) : x \leq y\}$$

   SOLUTION FOR PROBLEM 2.

   2.a) We find he pattern but omit the DFA

   We are in base 2. All $\equiv$ are mod 3.

   $2^0 \equiv 1$

   $2^1 \equiv 2$

   $2^2 \equiv 1$

   $2^3 \equiv 2$

   So that pattern is 1,2,1,2. These are the weights we use.

   2.b) We describe the DFA: as you scan in symbols of the form $00, 01, 10, 11$ (first bit is the x second is the y and we want $x \leq y$.

   There are two states: A (for Accept) and R for (for Reject). A is also the start state. The idea is that so long as the bits match or y looks

BIGGER than x stay in the A state. If you ever see a 10 (so x is BIGGER) goto R.

From R similar except that if ever y looks LESS THAN x we stay in R (a 10 comes in) , but if y looks BIGGER than x (a 01 came in) then goto A state.

$\delta(A, 00) = A$

$\delta(A, 01) = A$

$\delta(A, 10) = R$

$\delta(A, 11) = A$

$\delta(R, 00) = R$

$\delta(R, 01) = A$

$\delta(R, 10) = R$

$\delta(R, 11) = R$.

3. (30 points)

We want to write a DFA for:

$$\{(x_1, x_2, x_3, \ldots, x_n, y) : x_1 + \ldots x_n = y\}$$

(a) (5 points) If we add 3 numbers in base 2 (i.e. $\{(x_1, x_2, x_3, y) : x_1 + x_2 + x_3 = y\}$), what is the largest the carry can be?

(b) (5 points) If we add 4 numbers in base 2 what is the largest the carry can be?

(c) (5 points) If we add 5 numbers in base 2 what is the largest the carry can be?

(d) (5 points) If we add 6 numbers in base 2 what is the largest the carry can be?

(e) (5 points) Make a conjecture about the biggest carry when adding n numbers in base 2. (For Extra Credit - NOT for a grade but for future letters from me - PROVE your conjecture.)

(f) (5 points) If you were to write a DFA for:

$$\{(x_1, x_2, x_3, \ldots, x_n, y) : x_1 + \ldots x_n = y\}$$

how many states would it have? Explain.

SOLUTION TO PROBLEM 3

3.1)

If there are no carries then $1 + 1 + 1 + 1 = 100$ so the carry is $10 = 2$.

If you then add $2 + 1 + 1 + 1 + 1 = 6 = 110$ so the carry is $11 = 3$

If you then add $3 + 1 + 1 + 1 + 1 = 7 = 111$ so the carry is $11 = 3$

So the carry is at most 3.

3.2)

If there are no carries then $1+1+1+1+1 = 101$ so the carry is $10 = 2$.

If you then add $2 + 1 + 1 + 1 + 1 + 1 = 7 = 111$ so the carry is $11 = 3$

If you then add $3+1+1+1+1+1 = 8 = 1000$ so the carry is $100 = 4$

If you then add $4+1+1+1+1+1 = 8 = 1001$ so the carry is $100 = 4$

So the carry is at most 4.

3.3)

If there are no carries then $1 + 1 + 1 + 1 + 1 + 1 = 110$ so the carry is $11 = 3$.

If you then add $3 + 1 + 1 + 1 + 1 + 1 + 1 = 9 = 1001$ so the carry is $100 = 4$

If you then add $4 + 1 + 1 + 1 + 1 + 1 + 1 = \text{ten} = 1010$ so the carry is $101 = 5$

If you then add $5 + 1 + 1 + 1 + 1 + 1 + 1 = \text{eleven} = 1011$ so the carry is $101 = 5$

So the carry is at most 5.

3.4)

If there are no carries then $1 + 1 + 1 + 1 + 1 + 1 + 1 = 111$ so the carry is $11 = 3$.

If you then add $3 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = \text{ten} = 1010$ so the carry is $100 = 4$

If you then add $4 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = \text{eleven} = 1011$ so the carry is $101 = 5$

If you then add $5 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 12 = 1100$ so the carry is $1100 = 6$

If you then add $6 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 13 = 1101$ so the carry is $1100 = 6$

So the carry is at most 5.

3.5) n.

3.6) Here is the question:

If you were to do a DFA for

$$\{(x_1, \ldots, x_n, y) : x_1 + \cdots + x_n = y\}$$

how many states would it have? Explain.

There are $n + 2$ states. I need to store the previous carry so there are states $0, 1, \ldots, n$ for what the previous carry could be, AND a dump state.

4. (30 points) For each of the following state if it is REGULAR or NOT REGULAR. Prove your statement.

   (a) $\{a^n : \ n \text{ is a square number and } n \le 100\}$
   (b) $\{a^n : \ n \text{ is a square number and } n \ge 100\}$
   (c) $\{a^n a^n : n \in \mathsf{N}\}$
   (d) $\{w : \text{ There is a subword of } w \text{ that is a palindrome}\}$
   (e) $\{xyx^R : x, y \in \{a,b\}^*\}$

SOLUTION TO PROBLEM 5

5.1) FINITE so Regular.

5.2) Not Regular. Assume regular. Let n be large enough for the pumping lemma to hold. Then

$a^{n^2} = xyz$ where $x = a^{n_1}$, $y = a^{n_2}$, $z = a^{n_3}$.

By the pumping theorem, for all k, $xy^kz = a^{n_1+kn_2+n_3}$ is in L.

Let $n_1 + n_3 = m$.

The following are all squares:

$m + n_2 = (L+1)^2$ (I use $L+1$ so that later numbers work out nicely)

$m + 2n_2 \ge (L+2)^2$

$m + 3n_2 \ge (L+3)^2$

$m + 4n_2 \ge (L+4)^2$

$(\forall k)[m + kn_2 \ge (L+k)^2]$

This is

$(\forall k)[m + kn_2 \ge L^2 + 2Lk + k^2]$

Hence we have

$(\forall k)[m + kn_2 \ge k^2]$

Since k^2 grows faster than k and m, n_2 are constants this cannot hold for all k.

5.3) Regular: This is just $(aa)^*$.

5.4) Regular: This is just $\{a,b\} - \{e\}$ since a and b are palindromes.

5.5) Regular: This is just $\{a,b\}^*$. Note that if $x = e$ then all we get is y.