HW 10 CMSC 452. Morally Due April 17
SOLUTIONS

1. (5 points) What is your name? Write it clearly. Staple the HW.

2. (30 points) Show that the following problem is in NP:

   All (A, b) where

   (a) A is an $n \times n$ matrix of integers.

   (b) b is a vector of n integers

   (c) There exists a vector x of integers between -5 and 5 such that $Ax = b$

   (d) (Think about- not to be handed in) If we allow x to be a rational then is the problem still in NP? If we allow x to be any integer (so remove the bounds -5 to 5) then is the problem in NP?

   SOLUTION TO PROBLEM TWO

   The witness is just the vector x. The two things to check:

   - Is it easy to check that $Ax = b$. This takes $O(n^2)$ operations, so yes thats easy.
   - Is x short? Yes, x is n vectors of 3 bits each, so $O(n)$ length.

   If we let x be a rational OR if we let x be any integer than we would need to PROVE that if there is an x there is a short one. This is true but not obvious.

3. (30 points) If $G = (V, E)$ is a graph then $X \subseteq V$ is a *Vertex Cover* if for every $e \in E$, there is a $v \in X$ that is the endpoint of e.

   Show that the following problem is in P:

   $$\{G : \ G \text{ has a vertex cover of size 17 } \}$$

   SOLUTION TO PROBLEM THREE

   (a) Input G

(b) Look at all 17-sized sets of vertices X. There are $\le n^{17}$ of these. For each one CHECK if it is a vertex cover. This takes $O(e)$ time- for every edge check if one of the endpoints is in X. If ever you get a YES then output YES and stop

(c) If you got this far then output NO

The algorithm takes $O(en^17) \le O(n^{19})$.

4. (40 points) Describe an NFA with ≤ 500 states (it will actually be far less than this) for the set

$$\{a^y : (y \ne 999) \wedge (y \ne 1000)\}$$

Prove that it works by showing that if a^y is rejected then $y = 999$ or $y = 1000$.

HINT- For the big loop use 32 and 33. You may USE the fact that

- For all $n \ge 992$ there exists $x, y \in N$ such that $n = 32x + 33y$.
- There does not exist $x, y \in N$ such that $991 = 32x + 33y$.

SOLUTION TO PROBLEM FOUR

READ the writeup of the small NFA that only rejects a^{1000} before reading this.

Since 991 is not a linear combo of 32 and 33,

$991 - 33 = 958$, and $991 - 32 = 959$ are also not linear combos of 32 and 33.

Hence the big-loop NFA with a loop of size 33 and a shortcut for 32 does not accept a^{958} or a^{959}

Take that NFA and add 41 states at the beginning. NOW this NFA:

- For all $y \ge 1001$ accepts a^y.
- Does not accept a^{999}.
- Does not accept a^{1000}.

There are still many strings we want to accept but do not yet. Use e-transitions to add loops that recognize the following sets:

(a) $\{a^y : y \not\equiv 999 \pmod{3} \wedge y \not\equiv 1000 \pmod{3}\}$

(b) $\{a^y : y \not\equiv 999 \pmod{5} \wedge y \not\equiv 1000 \pmod{5}\}$

(c) $\{a^y : y \not\equiv 999 \pmod{7} \wedge y \not\equiv 1000 \pmod{7}\}$

(d) $\{a^y : y \not\equiv 999 \pmod{11} \wedge y \not\equiv 1000 \pmod{11}\}$

(e) $\{a^y : y \not\equiv 999 \pmod{13} \wedge y \not\equiv 1000 \pmod{13}\}$

(f) $\{a^y : y \not\equiv 999 \pmod{17} \wedge y \not\equiv 1000 \pmod{17}\}$

(g) $\{a^y : y \not\equiv 999 \pmod{19} \wedge y \not\equiv 1000 \pmod{19}\}$

Let M be the NFA that had from the start state e-transitions to the 33-loop-32-shortcut and to all the prime machines above. The number of states is

$$1 + 41 + 33 + 3 + 5 + 7 + 11 + 13 + 17 + 19 = 150$$

We show that the only strings REJECTED are a^{999} and a^{1000}.

Let a^y be a rejected string. Because of the big loops $y \le 1000$.

Because of the prime machines:

(a) $y \equiv 999 \pmod{3} \vee y \equiv 1000 \pmod{3}$

(b) $y \equiv 999 \pmod{5} \vee y \equiv 1000 \pmod{5}$

(c) $y \equiv 999 \pmod{7} \vee y \equiv 1000 \pmod{7}$

(d) $y \equiv 999 \pmod{11} \vee y \equiv 1000 \pmod{11}$

(e) $y \equiv 999 \pmod{13} \vee y \equiv 1000 \pmod{13}$

(f) $y \equiv 999 \pmod{17} \vee y \equiv 1000 \pmod{17}$

(g) $y \equiv 999 \pmod{19} \vee y \equiv 1000 \pmod{19}$

Take the set $X = \{3, 5, 7, 11, 13, 17, 19\}$.

For each $p \in X$ either $y \equiv 999 \pmod{p}$ OR $y \equiv 1000 \pmod{p}$.

Map p to either 999 or 1000 depending on which is the case.

Since X has 7 elements, either at least four map to 999 or at least 4 map to 1000.

Let those four be p_1, p_2, p_3, p_4.

If those four all map to 999 then

$y \equiv 999 \pmod p_1$

$y \equiv 999 \pmod p_2$

$y \equiv 999 \pmod p_3$

$y \equiv 999 \pmod p_4$

Hence

$y \equiv 999 \pmod p_1 p_2 p_3 p_4$.

since $p_1 p_2 p_3 p_4 \geq 3 \times 5 \times 7 \times 11 = 1155 > 999$ it must be that $y = 999$.

If those four all map to 1000 then

$y \equiv 1000 \pmod p_1$

$y \equiv 1000 \pmod p_2$

$y \equiv 1000 \pmod p_3$

$y \equiv 1000 \pmod p_4$

Hence

$y \equiv 1000 \pmod p_1 p_2 p_3 p_4$.

since $p_1 p_2 p_3 p_4 \geq 3 \times 5 \times 7 \times 11 = 1155 > 1000$ it must be that $y = 1000$.