The Roots Hierarchy

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1 Introduction

The main proof in this note is from *Problems from the Book* by Dospinescu and Andreescu.

We want to classify real numbers in terms of their complexity.

Def 1.1 Let $d \in \mathsf{N}$.

1. $\mathsf{Z}_d[x]$ is the set of polynomials of degree d over Z (the integers).

2. roots_d is the set of roots of polynomials in $\mathsf{Z}_d[x]$. Note that $\mathrm{roots}_1 = \mathsf{Q}$.

Clearly $\mathrm{roots}_1 \subseteq \mathrm{roots}_2 \subseteq \mathrm{roots}_3 \subseteq \cdots$

We want to show that $\mathrm{roots}_1 \subset \mathrm{roots}_2 \subset \mathrm{roots}_3 \subset \cdots$

2 The Hierarchy is Proper

We show that $\mathrm{roots}_3 \subset \mathrm{roots}_4$. All of the ideas to show $\mathrm{roots}_{d-1} \subset \mathrm{roots}_d$ are contained in the proof. The main method for the proof is taken from chapter 9 of *Problems from the Book* by Titu Andreescu and Gabriel Dospinescu.

Theorem 2.1 $\mathrm{roots}_3 \subset \mathrm{roots}_4$.

Proof: Clearly $\mathrm{roots}_3 \subseteq \mathrm{roots}_4$. We show that $2^{1/4} \in \mathrm{roots}_4 - \mathrm{roots}_3$ which implies

$$\mathrm{roots}_3 \subset \mathrm{roots}_4.$$

Clearly $2^{1/4}$ is a root of $x^4 - 2 = 0$ and hence $2^{1/4} \in \mathrm{roots}_4$. We show that $2^{1/4} \notin \mathrm{roots}_3$

Assume, by way of contradiction, that there exists $a_0, a_1, a_2, a_3 \in \mathsf{Z}$ such that

$$a_3(2^{1/4})^3 + a_2(2^{1/4})^2 + a_1(2^{1/4}) + a_0 = 0$$

which is

$$a_3 \times 2^{3/4} + a_2 \times 2^{1/2} + a_1 \times 2^{1/4} + a_0 \times 1 = 0$$

We assume the following about (a_3, a_2, a_1, a_0): *They are not all even.* If they are then divide each one by 2 to get a smaller poly over Z and use that.

Multiply this equation by 1, $2^{1/4}, 2^{1/2}, 2^{3/4}$ to get

$$a_3 \times 2^{3/4} + a_2 \times 2^{1/2} + a_1 \times 2^{1/4} + a_0 \times 1 = 0$$

$$a_2 \times 2^{3/4} + a_1 \times 2^{1/2} + a_0 \times 2^{1/4} + 2a_3 \times 1 = 0$$

$$a_1 \times 2^{3/4} + a_0 \times 2^{1/2} + 2a_3 \times 2^{1/4} + 2a_2 \times 1 = 0$$

$$a_0 \times 2^{3/4} + 2a_3 \times 2^{1/2} + 2a_2 \times 2^{1/4} + 2a_1 \times 1 = 0$$

We rewrite this as a matrix times a vector being the zero vector:

$$\begin{pmatrix} a_3 & a_2 & a_1 & a_0 \\ a_2 & a_1 & a_0 & 2a_3 \\ a_1 & a_0 & 2a_3 & 2a_2 \\ a_0 & 2a_3 & 2a_2 & 2a_1 \end{pmatrix} \begin{pmatrix} 2^{3/4} \\ 2^{1/2} \\ 2^{1/4} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Let

$$A = \begin{pmatrix} a_3 & a_2 & a_1 & a_0 \\ a_2 & a_1 & a_0 & 2a_3 \\ a_1 & a_0 & 2a_3 & 2a_2 \\ a_0 & 2a_3 & 2a_2 & 2a_1 \end{pmatrix}$$

The matrix above can be multiplied by a non-zero vector and get zero. Hence the matrix has det 0. Hence the det is 0 MOD 2.

$$A \pmod 2 = \begin{pmatrix} a_3 & a_2 & a_1 & a_0 \\ a_2 & a_1 & a_0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_0 & 0 & 0 & 0 \end{pmatrix}$$

By the column expansion definition of det, applied to the last row the det (mod 2) is a_0^4, Hence $a_0^4 \equiv 0 \pmod 2$, so $a_0 \equiv 0 \pmod 2$. We rewrite A:

$$A = \begin{pmatrix} a_3 & a_2 & a_1 & 2b_0 \\ a_2 & a_1 & 2b_0 & 2a_3 \\ a_1 & 2b_0 & 2a_3 & 2a_2 \\ 2b_0 & 2a_3 & 2a_2 & 2a_1 \end{pmatrix}$$

Since this matrix had det 0, so does the matrix when I divide the last column by 2. Hence this matrix has det 0:

$$B = \begin{pmatrix} a_3 & a_2 & a_1 & b_0 \\ a_2 & a_1 & 2b_0 & a_3 \\ a_1 & 2b_0 & 2a_3 & a_2 \\ 2b_0 & 2a_3 & 2a_2 & a_1 \end{pmatrix}$$

Take this matrix mod 2 to get:

$$B \pmod 2 = \begin{pmatrix} a_3 & a_2 & a_1 & b_0 \\ a_2 & a_1 & 0 & a_3 \\ a_1 & 0 & 0 & a_2 \\ 0 & 0 & 0 & a_1 \end{pmatrix}$$

If you expand the det of $B \pmod 2$ on the last row you get a_1^4. Hence $a_1^4 \equiv 0 \pmod 2$, so $a_1 \equiv 0 \pmod 2$. Hence $a_1 = 2b_1$. We rewrite B:

$$B = \begin{pmatrix} a_3 & a_2 & 2b_1 & b_0 \\ a_2 & 2b_1 & 2b_0 & a_3 \\ 2b_1 & 2b_0 & 2a_3 & a_2 \\ 2b_0 & 2a_3 & 2a_2 & 2b_1 \end{pmatrix}$$

We divide the third column by 2:

$$C = \begin{pmatrix} a_3 & a_2 & b_1 & b_0 \\ a_2 & 2b_1 & b_0 & a_3 \\ 2b_1 & 2b_0 & a_3 & a_2 \\ 2b_0 & 2a_3 & a_2 & 2b_1 \end{pmatrix}$$

Hence

$$C \pmod 2 = \begin{pmatrix} a_3 & a_2 & b_1 & b_0 \\ a_2 & 0 & b_0 & a_3 \\ 0 & 0 & a_3 & a_2 \\ 0 & 0 & a_2 & 0 \end{pmatrix}$$

By the column expansion definition of det, applied to the last row, a_2 is even. Let

$a_2 = 2b_2$. If a matrix has det 0 and you divide a column by (say) 2 then the matrix still has det 0. Divide the second column by 2, and replace all a_2 by $2b_2$, to get:

$$D = \begin{pmatrix} a_3 & b_2 & b_1 & b_0 \\ 2b_2 & b_1 & b_0 & a_3 \\ 2b_1 & b_0 & a_3 & 2b_2 \\ 2b_0 & a_3 & 2b_2 & 2b_1 \end{pmatrix}$$

$$D \pmod 2 = \begin{pmatrix} a_3 & b_2 & b_1 & b_0 \\ 0 & b_1 & b_0 & a_3 \\ 0 & b_0 & a_3 & 0 \\ 0 & a_3 & 0 & 0 \end{pmatrix}$$

By the column expansion definition of det, applied to the last column, a_3 is even.

We now have that a_3, a_2, a_1, a_0 are all even. This contradicts are assumption on (a_3, a_2, a_1, a_0).

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