HW 7 CMSC 452. Morally Due April 16
THIS HW IS TWO PAGES LONG!!!!!!!!!
SOLUTIONS

1. (30 points) Let

$$IS = \{(G,k) \mid \text{ graph } G \text{ has an independent set of size } k \ \}$$

   (a) (10 points) Show that $CNF\text{-}SAT \le IS$. Do not use CLIQUE as an intermediary. Explain why your reduction works.

   (b) (10 points) On the formula:

$$(x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2) \wedge \neg x_3$$

   what (G,k) does your reduction produce? Draw the graph.

   (c) (10 points) On the formula:

$$(x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2) \wedge x_4$$

   what (G,k) does your reduction produce? Draw the graph.

SOLUTION TO PROBLEM ONE

Given a formula $\phi = C_1 \wedge \cdots \wedge C_k$ where each C_i is an OR of literals we produce (G,k). k is already revealed: it is the number of clauses.

For each C_i have a group of vertices labeled with the literals in C_i (formally labeled with an i as well to say where it came from).

Add an edge between each pair of vertices in the SAME group. Also add an edge between two vertices of DIFFERENT groups if they DO contradict each other.

If G has a k-IS then it has a literal in every clause that DOES NOT contradict the other literals in the IS since there are no edges between them. Hence $\phi \in SAT$ since we can set all of those literals true at the same time. In brief: If $(G,k) \in IS$ then $\phi \in SAT$.

If $\phi \in SAT$ then look at a satisfying assignment. It will have one true literal in each clause. This set of literals forms a k-IS. In brief $\phi \in SAT$ iff $(G,k) \in IS$.

GO TO THE NEXT PAGE!!!!!!!!!!!!!!!!!!!!

2. (35 points) Let

$$\text{CLIQ} = \{(G, k) \mid \text{ graph } G \text{ has a clique of size } k\ \}$$

Let

SCLIQ be the FUNCTION that will, on input G, output the SIZE of the largest clique.

Let

FINDCLIQ be the FUNCTION that will, on input G, output both the SIZE of the largest clique, and SOME clique of that size (that is, a list of vertices that form a clique of max size).

(a) (17 points) Show that if CLIQ $\in P$ then SCLIQ can be computed in polynomial time. (ADDED LATER TO CLARIFY: The *CLIQ TM* is the Poly Time TM that decides the set CLIQ.) (THINK ABOUT but don't hand in: Your algorithm made several calls to the CLIQ TM. How many? Assuming $P \neq NP$, is there a way to do with this with 18 calls to CLIQ?)

(b) (18 points) Show that if CLIQ $\in P$ then FINDCLIQ can be computed in polynomial time. (THINK ABOUT but don't hand in: Your algorithm made several calls to the CLIQ TM. How many? Assuming $P \neq NP$, is there a way to do with this with 18 calls to CLIQ?)

SOLUTION TO PROBLEM TWO

1) Assume CLIQ $\in P$. Then given G we find SCLIQ as follows. Let n be the number of vertices in G. First ask $(G, n/2) \in$ CLIQ. Then proceed by binary search to find the k such that $(G, k) \in$ CLIQ but $(G, k+1) \notin$ CLIQ. Output k. This took $O(\log n)$ calls to CLIQ.

2) Assume CLIQ $\in P$. Then given G we find FINDCLIQ as follows. Let n be the number of vertices in G. First use the answer to part 1 to find the max clique size which we call k. Let the vertices be $1, \ldots, n$.

Ask $(G - \{n\}, k) \in$ CLIQ

If NO then great, we know that there is a clique of size k that includes n. Put n aside for now, it will be in the final clique. Hence we can look at just the neighbors of n. Call the graph of neighbors of n, without n, G'. We seek a clique of size $k-1$ of G'. Repeat the procedure (though read on to see what you do if the answer was YES)

If YES then great, we know that there is a k-clique that does not include n. So toss n out. Keep doing this until you get a NO and are in the other category.

The number of calls to CLIQ is $O(\log n)$ to find the max size of a clique, and then you ask about $G - i$ for each i at most once, so $O(n)$ calls. So $O(n) + O(\log n)$ calls to CLIQ in total.

3. (35 points) Let

$$3COL = \{G : \text{ graph } G \text{ is 3-colorable}\}.$$

Show that $3COL \le SAT$. Give an explicit reduction and explain why it works.

(HINT: Let G have vertices $1, \ldots, n$. The variables are, for $1 \le i \le 3$, $1 \le j \le n$, x_{ij}. The variable x_{ij} is intended to be TRUE if Vertex j is colored i, and FALSE if not.)

SOLUTION TO PROBLEM THREE

We give the formula in parts that are AND-ed together.

CLAUSES THAT MAKE SURE x_{ij} MAKE SENSE:

(a) For all $1 \le j \le n$, $x_{1j} \vee x_{2j} \vee x_{3j}$. Hence every node gets AT LEAST one color.

(b) For all $1 \le j \le n$, j CANNOT have both color 1 and 2: $\neg x_{1j} \vee \neg x_{2j}$

(c) For all $1 \le j \le n$, j CANNOT have both color 1 and 3: $\neg x_{1j} \vee \neg x_{3j}$

(d) For all $1 \le j \le n$, j CANNOT have both color 2 and 3: $\neg x_{2j} \vee \neg x_{3j}$

And NOW the clauses that make sure its a proper 3-coloring:

For each edge (a, b),

$$\neg(x_{1a} \wedge x_{1b}) \wedge \neg(x_{2a} \wedge x_{2b}) \wedge \neg(x_{3a} \wedge x_{3b})$$