

HW 9 CMSC 452. Morally Due April 30
THIS HW IS TWO PAGES LONG!!!!!!!!!!
SOLUTIONS

Throughout this HW M_1, M_2, \dots is a standard list of Turing Machines. Can also view as a list of all partial computable functions.

1. (30 points) Answer TRUE OR FALSE and prove it. Assume input and output take values in \mathbb{N} .
 - (a) (15 points) Let f be a computable function such that for all $x < y$, $f(x) < f(y)$. Then the IMAGE of f is computable. (Recall that the IMAGE of f is

$$\{y \mid (\exists x)[f(x) = y]\}.$$

)

- (b) (15 points) Let f be a computable function such that for all $x < y$, $f(x) \leq f(y)$. Then the IMAGE of f is computable.
- (c) (0 points — but think about) Let f be a computable function such that for all $x < y$, $f(x) > f(y)$. Then the IMAGE of f is computable.
- (d) (0 points — but think about) Let f be a computable function such that for all $x < y$, $f(x) \geq f(y)$. Then the IMAGE of f is computable.

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SOLUTION TO PROBLEM ONE

- (a) Let f be a computable function such that for all $x < y$, $f(x) < f(y)$. Then the IMAGE of f is computable: TRUE

Here is the algorithm.

- i. Input y
- ii. Compute $f(0), f(1), \dots$ until EITHER
 - You find an x such that $f(x) = y$ — in which case output YES, or
 - You find an x such that $f(x) < y$ but $f(x + 1) > y$ — in which case output NO

KEY-One of these two MUST happen since f is strictly increasing.

- (b) Let f be a computable function such that for all $x < y$, $f(x) \leq f(y)$. Then the IMAGE of f is computable: TRUE

The proof is NONCONSTRUCTIVE.

There are two cases:

Case 1: f is unbounded. Then the image of f is computable using the algorithm in part 1.

Case 2: f is eventually constant. The image of f is FINITE and hence computable.

- (c) Let f be a computable function such that for all $x < y$, $f(x) > f(y)$. Then the IMAGE of f is computable: TRUE

There are no such functions. If $f(0) = n$ then $f(1) \leq n - 1, \dots, f(n) \leq 0, f(n + 1) \leq -1$, but f has codomain \mathbf{N} .

Hence the statement is true vacuously. Or, to put it another way, to show its false you would need to GIVE ME a computable f from \mathbf{N} to \mathbf{N} with IMAGE not computable. You can't do this since you can't give me such an f .

- (d) Let f be a computable function such that for all $x < y$, $f(x) \geq f(y)$. Then the IMAGE of f is computable: TRUE

If $f(0) = n$ then the image of f is a subset of $\{0, \dots, n\}$ and hence finite and hence computable.

2. (30 points) Let *YAEILLE* be the set of functions from \mathbf{N} to \mathbf{N} generated by the following properties:

- Every primitive recursive function on one variable is in *YAEELLE*.
- Recall that Ackermann's function was on two variables. Let $A(x, y)$ be Ackermann's function. Then $f(x) = A(x, x)$ is in *YAEELLE*.
- If $f, g \in YAEELLE$ then $f(g(x))$ is in *YAEELLE*.

An now we need a definition:

A function g DOMINATES a function f if, for all but a finite number of x , $f(x) < g(x)$.

(Note that if p_i dominates p_j ($p_i(x) > p_j(x)$ on all but a finite number of inputs), then there is SOME x such that $p_i(y) > p_j(y)$ for all $y > x$.)

And NOW finally the question:

- (15 points) Show that there exists a COMPUTABLE function g that DOMINATES all *YAEELLE* functions.
- (15 points) Show that if a function f DOMINATES all *YAEELLE* functions, then f is NOT a *YAEELLE* function.

SOLUTION TO PROBLEM TWO

1) The KEY is that you can LIST out the functions in *YAEELLE* in some reasonable way AND they are all total. Let

$$f_1, f_2, \dots$$

be all of the function in *YAEELLE*.

Then let

$$g(x) = \max\{f_1(x), f_2(x), \dots, f_x(x)\} + 1.$$

Let f be *YAEELLE*. Then f is some f_i . Then

For all $x \geq i$, $f_i(x) < g(x)$ since $g(x) = \max\{\dots, f_i(x), \dots\} + 1$.

2) Assume, by way of contradiction, that g dominates every *YAEELLE* function and g is *YAEELLE*. Then g dominates itself. Hence, there is an x (many actually) such that $g(x) < g(x)$. This is a contradiction.

3. (40 points) Let p_1, p_2, \dots be all the primitive recursive functions.

For each of the following sets WRITE IT using quantifiers and try to get it as low as possible in the arithmetic hierarchy (i.e., given set X try to find the least $n \in \mathbb{N}$ such that $X \in \Sigma_n$ or $X \in \Pi_n$). STATE for each one where it is (e.g., $X \in \Sigma_{100}$).

(a) (10 points)

$$A = \{(i, j) \mid p_i \text{ and } p_j \text{ are the same function}\}$$

(b) (10 points)

$$B = \{(i, j) \mid p_i \text{ dominates } p_j\}$$

(c) (10 points)

$$C = \{i \mid M_i \text{ halts on all but at most 2 inputs}\}$$

(d) (10 points)

$$D = \{i \mid M_i \text{ halts on all but a finite number of inputs}\}$$

SOLUTION TO PROBLEM THREE

(a)

$$A = \{(i, j) \mid p_i \text{ and } p_j \text{ are the same function}\} = \\ \{(i, j) \mid (\forall x)[p_i(x) = p_j(x)]\}$$

So $A \in \Pi_1$.

(b)

$$B = \{(i, j) \mid p_i \text{ dominates } p_j\} = \\ \{(i, j) \mid (\exists x)(\forall y)[y > x \rightarrow p_i(y) > p_j(y)]\}$$

So $B \in \Sigma_2$.

(c)

$$C = \{i \mid M_i \text{ halts on all but at most 2 inputs}\} = \\ \{i \mid (\exists x_1, x_2)(\forall y)(\exists s)[(y \neq x_1 \wedge y \neq x_2) \rightarrow M_{i,s}(y) \downarrow]\}$$

So $C \in \Sigma_3$.

- (d) Note that if M_i halts on all but a finite number of inputs then past SOME point it always converges.

$$D = \{i \mid M_i \text{ halts on all but a finite number of inputs}\} =$$

$$\{i \mid (\exists x)(\forall y)(\exists s)[y \geq x \rightarrow M_{i,s}(y) \downarrow]\}$$

So $D \in \Sigma_3$.