452 MIDTERM

Do not open this exam until you are told. Read these instructions:

1. This is a closed book exam, though ONE sheet of notes is allowed. **No calculators, or other aids are allowed**. If you have a question during the exam, please raise your hand.
2. There are 5 problems which add up to 100 points. The exam is 1 hours 15 minutes. (You shouldn't need that much.)
3. For each question show all of your work and **write legibly**. **Clearly indicate** your answers. No credit for illegible answers.
4. After the last page there is paper for scratch work. If you need extra scratch paper **after** you have filled these areas up, please raise your hand. Scratch paper must be turned in with your exam, with your name and ID number written on it, but scratch paper **will not** be graded.
5. Please write out the following statement: "*I pledge on my honor that I will not give or receive any unauthorized assistance on this examination.*"

Fill in the following:

NAME :
SIGNATURE :
SID :
SECTION NUMBER :

1. (20 points) For each of the following languages say if it's REGULAR or NOT REGULAR. If REGULAR then draw a DFA or give a regular expression for it. (NOTE — I REALLY WANT the DFA or REGULAR expression for it!!!!!!!!!!!!!!!!) If NOT REGULAR then prove it is not regular.

   The alphabet is $\{a, b\}$ and $\mathsf{N} = \{0, 1, 2, \ldots\}$.

   (a)
   $$L_1 = \{a^n b^m \mid n, m \in \mathsf{N}\}$$

   (b)
   $$L_2 = \{a, b\}^* - L_1.$$

   (In other words, all strings that are NOT in L_1.) You may use the answer to part a in this part.

   (c)
   $$L_3 = \{a^n b^m \mid n > 2m\}$$

   You may use the answer to part a or b in this part.

   (d)
   $$L_4 = \{a, b\}^* - L_3.$$

   (In other words, all strings that are NOT in L_3.) You may use the answer to part a or b or c in this part.

SOLUTION TO PROBLEM ONE

GRADING: 2 points for getting REG vs NOT-REG correct. 3 for justification. For those 3 everyone got 0 or 3.

(a)

$$L_1 = \{a^n b^m \mid n, m \in \mathsf{N}\}$$

REGULAR. This is just $a^* b^*$.

(b)

$$L_2 = \{a, b\}^* - L_1.$$

(In other words, all strings that are NOT in L_1.) (If you prove that L_1 is not regular you may use that in this part.)

REGULAR: You CANNOT just say REG closed under Complement. I said I REALLY want the DFA or REGEX for it. Here is one possible regular expression: $\{a, b\}^* ba \{a, b\}^*$. Easier: just do a DFA for it!

NOTE: I warned you that I wanted a Regular Expression, so if you used complementation you lost 3 points.

(c)

$$L_3 = \{a^n b^m \mid n \geq 2m\}$$

NOT REGULAR.

Let $w = a^n b^{2n}$. By extended Pumping Lemma we can make sure the y is within the a's So:

$w = xyz$.

$x = a^{n_1}$, $y = a^{n_2}$, $z = a^{n_3} b^{2n}$ where:

- $n_2 \neq 0$ (since $y \neq e$)
- $n_1 + n_2 + n_3 = n$

We know that $\forall i \geq 0$, $xy^i z \in L$. Much like on the solution to HW5, problem 2e, the posted solutions, we pump ZERO times to get

$$a^{n_1+n_3} b^{2n}$$

and note that since $n_2 \neq 0$, $n_1 + n_3 < n_1 + n_2 + n_3 < n$, so NOT in L.

NOTE: When using Pumping Lemma you CANNOT choose the x, y, z with the exception of the Extended Pumping lemma when you choose a bit- like that y is in the a's. Some people tried to claim that x must be e or that y must be a's AND b's. None of this work.

(d)

$$L_4 = \{a, b\}^* - L_3.$$

(In other words, all strings that are NOT in L_3.) (If you prove that L_3 is not regular you may use that in this part.)

NOT REGULAR: If L_4 was regular than, since regular is closed under Comp, L_3 would be regular.

NOTE: If you just said *since L_3 is not regular and the exam says we should use that for this part, L_4 is not regular* then you got ZERO. Of course nobody quite said that but thats what they meant when they said *This follows from L_3 being not regular* without mentioning complementation. Or similar statements that allude to L_3 not being regular but do not mention complementation.

2. (20 points)

Using WS1S notation let:

$$L = \{(A, B) \mid A \cap B = \emptyset\}$$

(a) Draw a DFA, with each state labeled S (for Stupid), A (for Accept), or R (for Reject) for L.

(b) Give a Regular Expression for L. (HINT: do NOT use the DFA to REGEX $R(i, j, k)$ conversion.)

SOLUTION TO PROBLEM TWO omitted.

NOTE: If you swapped ACCEPT and REJECT states then you lost 5 points (out of 20).

We allowed the initial state to be stupid since it was not clear if the empty string is the empty set (it is, but we never said so in class). However, if you loop BACK to a stupid state, you lose 5 points.

3. (20 points) This problem uses the WS1S conventions for format.

   (a) Draw a DFA, with each state labeled S (for Stupid), A (for Accept), or R (for Reject) for the following language:
   $\{(x, y) \mid x \le y \text{ AND } x \equiv y \pmod 3\}$

   (b) Let $n \in \mathsf{N}$ and $n \ge 3$. Draw a DFA, with each state labeled S (for Stupid), A (for Accept), or R (for Reject) for the following language:
   $\{(x, y) \mid x \le y \text{ AND } x \equiv y \pmod n\}$
   (you may use DOT DOT DOT)
   (ADVICE- DO NOT draw TWO DFA's and do cross product- that will just be a mess.)

SOLUTION TO PROBLEM THREE omitted.

Some students had e-transitions in their DFA. Even for NDFA's for WS1S this makes no sense.

4. (20 points) Let $\Sigma = \{a, b\}$. L_1 be regular via DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$, and L_2 be regular via DFA $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$.

Note that \$ is NOT in Σ. Construct a DFA for:

$$\{x\$y \mid x \in L_1 \text{ and } y \notin L_2\}$$

using M_1 and M_2. Briefly describe how your DFA is constructed. (NOTE- we are asking for a construction, not a drawing. Recall we did constructions with DFA's to prove that if L_1 and L_2 are regular than so are BLAH.)

NOTE- Do NOT give an NFA and say *we can convert it to a DFA*. I want the DFA!

REMINDER: Your DFA must have, for every state q and alphabet symbol σ, $\delta(q, \sigma)$ defined.

SOLUTION TO PROBLEM FOUR

$(Q_1 \cup Q_2 \cup \{\text{DUMP}\}, \Sigma \cup \{\$\}, \delta, s_1, Q_2 - F_2)$

Where δ is defined as follows:

If $q \in Q_1$ and $\sigma \in \{a, b\}$ then $\delta(q, \sigma) = \delta_1(q, \sigma)$.

If $q \in Q_1 - F_1$ then $\delta(q, \$) = \text{DUMP}$

If $q \in F_1$ then $\delta(q, \$) = s_2$

If $q \in Q_2$ and $\sigma \in \{a, b\}$ then $\delta(q, \sigma) = \delta_2(q, \sigma)$.

If $q \in Q_2$ then $\delta(q, \$) = \text{DUMP}$.

$\delta(\text{DUMP}, \sigma) = \text{DUMP}$ for $\sigma \in \{a, b, \$\}$.

NOTES:

- Some students tried to do a cross product construction. I suspect they did the one thing I told students $\aleph_0$ times: *do not memorize things* and *not copy things off of your cheat sheet*. These students got 0.
- There was a mild ambiguity in the problem. We had $y \notin L_2$. Do we mean $\{a, b\}^* - L_2$ or $\{a, b, \$\}^* - L_2$. We were happy with either interpretation. Almost everyone interpreted it as $\{a, b\}^* - L_2$.

- Some students did not have a dead state. We let this one pass. We will not be so generous in the future. Why did you leave it out when I said explicitly that every $\delta(q, \sigma)$ has to be defined? These are the questions that try ones soul.
- Some students copied the L_1L_2 construction, so they had e instead of \$ for going from M_1 to M_2. Note that this is NOT a DFA. And they are likely copying from their cheat sheets. They got ZERO.

5. (20 points) For this problem you may use the following theorem.

   Theorem: If x, y are relatively prime then

   - For all $z \ge xy - x - y + 1$ there exists $c, d \in \mathsf{N}$ such that $z = cx + dy$.
   - There is no $c, d \in \mathsf{N}$ such that $xy - x - y = cx + dy$.

   The alphabet is $\{a\}$. Let

$$L = \{a^n \mid n \neq 117\}$$

   (a) (10 points) Does there exist a DFA for L with less than 120 states? If so then draw the DFA; you may use DOT DOT DOT (You DO NOT have to prove that it works.) If not then PROVE there is no such DFA. (The 120 is NOT a typo. We really do mean 120.)

   (b) (10 points) Does there exist an NFA for L with less than 60 states? If so then draw the NFA; you may use DOT DOT DOT (You DO NOT have to prove that it works.) If not then PROVE there is no such NFA.

SOLUTION TO PROBLEM FIVE

There were two versions of the exam. One version had problem 5 with 117 and one with 202. We give the answer for 202. We make a comment about 117 later.

Part 1 YES. We omit the DFA but its long and skinny DFA and has $201 < 202$ states.

NOTE: If a student wrote that there was NO such DFA and proceeded to (likely by memorization without any real understanding) prove it (incorrectly) they got 0 points. Again, do not memorize, you must understand!

NOTE: If a student wrote YES there is such a DFA and either didn't present a DFA or presented one that looks nothing like the right one, then they got ZERO. If it loops back to the start state, resembling a HW and perhaps copied off of your cheat sheet, you get 5 out of the 10.

Part 2: YES. Use that

If x, y are rel prime then $xy - x - y$ cannot be written as the sum of x's and y's, but any number larger can be.

We need to find x, y such that $xy - x - y$ is just below 200

$x = 14$ and $y = 15$. Then $xy - x - y = 181$.

SO

- 200 CANNOT be written as $19 + 14a + 15b$.
- Every number > 200 can be written as $19 + 14a + 15b$.

- From the start state to a point you'll have a loop, have 19 states. This takes 19 state.
- Then have a loop of size 15 with a one-edge 14 shortcut. This takes care of all $n \ge 201$. This is only 15 states
- Also have an e-transition to a DFA for $n \equiv 1 \bmod 2$.
- Also have an e-transition to a DFA for $n \equiv 0, 1 \bmod 3$.
- Also have an e-transition to a DFA for $n \equiv 1, 2, 3, 4 \bmod 5$.
- Also have an e-transition to a DFA for $n \equiv 0, 1, 2, 3, 5, 6 \bmod 7$.

NOTES:

- If a student just wrote YES (there is such an NFA) and nothing else, that was worth 0.
- For $n = 117$ one can use $x = 10$ and $y = 13$ to get

$$xy - x - y = 130 - 10 - 23 = 107$$

  which is close to 117.

Scratch Paper

stuff